- Now that we have explored Maxwell's Equations in full detail, it is important to understand a key feature of their nature – electromagnetic waves.
- We will show (in a bit) that under certain condition Maxwell's Equations give rise to wave solutions.

Waves

- Waves are a travelling disturbance

eg. water waves are a disturbance of height, sound waves are a disturbance of pressure, waves in a wheat field are disturbances of positions of the stalks.

- A "1D" wave refers (generally) to the movement being 1D, so a taut ~~wave~~ string is 1D: $f(x,t)$

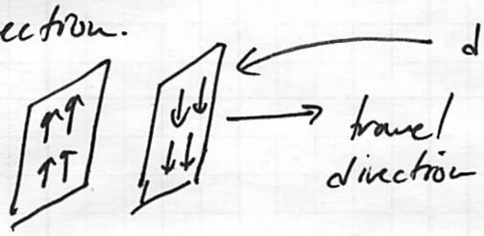
measure of the disturbance

1D ⇒ travels in only $\pm x$.

- Now the string might wiggle in y or $\cancel{z}$, that'd be f, but it only travels in x, down the string, so that's 1D.

- A "3D" wave propagates in 3D-space. It <u>might</u> still travel in a straight line, but that line is a vector in 3D space of possible directions. (sound waves typically spread out in all directions.)

- A "plane wave" is a 3D wave that travels in one direction, in 3D space, and the disturbance is identical for all points in the infinite plane transverse in that direction.



displacement is same at all points in any plane ⊥ to travel direction.

(Far from source, soundwaves look like plane waves.)

The Wave Equation

In 1D simple waves obey the PDE,

$$\frac{\partial^2 f(z,t)}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \qquad \text{(1D waves in z-direction)}$$

[Griffiths derives this for a taut string, but many systems can be modeled this way]

Claim: Given any 1D function $g(z)$, then $f(z,t) = g(z-vt)$ solves the wave eqn. Let's see how, $u = z - vt$

$$\frac{\partial f}{\partial z} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial z} = \frac{\partial g}{\partial u} \qquad\qquad \frac{\partial f}{\partial t} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial t} = -v\frac{\partial g}{\partial u}$$

so that,

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial u}\right) = \frac{\partial^2 g}{\partial u^2}\frac{\partial u}{\partial z} = \frac{\partial^2 g}{\partial u^2} \qquad \frac{\partial^2 f}{\partial t^2} = -v\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial u}\right) = -v\frac{\partial^2 g}{\partial u^2}\frac{\partial u}{\partial t} = v^2\frac{\partial^2 g}{\partial u^2}$$

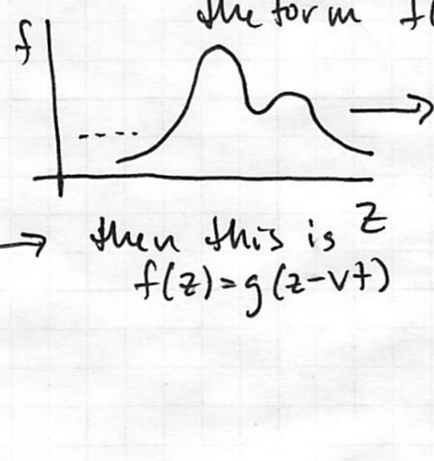
so,

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 g}{\partial u^2} \quad \text{and} \quad \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 g}{\partial u^2}$$

they must be equal as $\partial^2 g/\partial u^2 = \partial^2 g/\partial u^2$ so,

$$\frac{\partial^2 g}{\partial u^2} = \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

So solutions must take the form $f(z,t) = g(z-vt)$? we will see.



this might be $g(z)$ → then this is $f(z) = g(z-vt)$ } it's just $g(z)$ moving to the right at speed v.

Claim! $f(z,t) = g(z+vt)$ also solves the wave eqn.

You can try this yourself, but notice only v^2 appears in the wave eqn. So the same shape but now it travels to the left!

Claim! The fully general solution can always be written as some $g(z-vt) + h(z+vt)$

Note! you can generate solutions that aren't so obvious in this way.

For example, Let $g(z) = h(z) = z$ so $f(z,t) = 2z$

this solution doesn't travel but satisfies wave eqn.

Let $g(z) = h(z) = \cos z$, so $f(z,t) = \cos(z-vt) + \cos(z+vt)$
$$= 2\cos(z)\cos(vt)$$

this wiggles but doesn't travel $\Rightarrow$ standing wave!

There are many (∞) solutions to this PDE. But we will focus our attention on one elegant, useful solution:

The sinusoidal travelling wave

$$f(z,t) = A\cos\left(k(z-vt) + \delta\right)$$

We can rewrite with $kv = \omega$ as,

$$f(z,t) = A\cos\left(kz - \omega t + \delta\right)$$

Definitely a solution b/c form is $g(z-vt)$

if a rightward moving solution if $v > 0$.

Sinusoidal Waves

$$f(z,t) = A\cos(kz - \omega t + \delta)$$

Fourier tell us that we can build up complicated solutions by summing sinusoidal solutions with different k's, A's (and ω's). B/c the PDE is linear, these solutions satisfy the wave equation. So ~~[illegible]~~ while sinusoidal solutions keep things relatively simple, they are quite powerful.

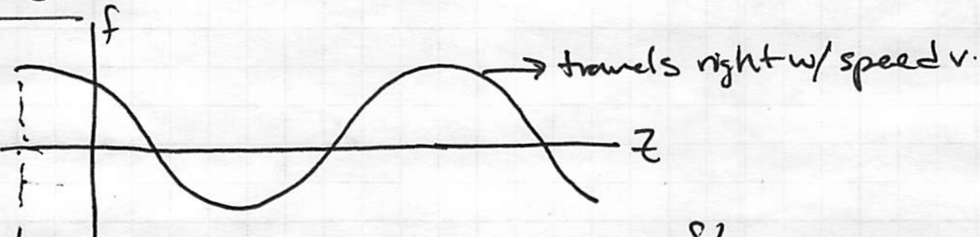
- Our sinusoidal solution has a definite periodicity in z, called the wavelength, $\lambda = 2\pi/k$

Also a definite period, $T = 2\pi/\omega$ and thus a definite frequency $f = 1/T$ (Recall $f\lambda = v$)

[Note: $\omega = 2\pi f = kv$ is called the angular frequency]

Easy to visualize

At $t=0$



↳ peak is shifted left by $\Delta z = -\delta/k$ at $t=0$ so it's "phase delayed" by δ.

Bad News: Our solution is ∞ in its extent, it has no beginning or end or edge.

If you are picturing a single "pulse", you may not be thinking about the wave correctly.

[Note: we could have use sin instead of cos, it's just convention to use cos. (just changes δ actually)]

<u>More Conventions</u> : for left moving waves, we choose to write them as:

$$f_{left}(z,t) = A\cos\left(k(z+vt) - \delta\right)$$

note: we switch the sign there.

$$= A\cos\left(kz+\omega t - \delta\right)$$

Why switch the sign on δ? So the wave is also <u>delayed</u> at $t=0$ by δ/k. The wave is left moving!

Note: If $f_{right}(z,t) \equiv A\cos(kz - \omega t + \delta)$

and $f_{left}(z,t) \equiv A\cos(kz+\omega t - \delta) = A\cos(-kz-\omega t+\delta)$

b/c $\cos(-x) = \cos(x)$

<u>So we can obtain f_{left} from f_{right} by taking $k \to -k$!</u>

<u>Convention</u> : when we write $f(z,t) = A\cos(kz-\omega t+\delta)$, we should think:

- "ω is always positive" (by convention)
- "if $\delta > 0$ the wave is delayed"
- "if $k>0$, the wave moves right. if $k<0$, it moves left."

So convention is that $\omega > 0$ but k can flip signs.

<u>Note:</u> $|k|$ is the wavenumber, $|A|$ is the amplitude

δ is the phase shift, T is the period $= 2\pi/\omega$

$|v|$ is the wave speed $= f\lambda$

But v is also ω/k (sign of v = sign of k!)

$+k$ means $+v$ right moving wave

$-k$ means $-v$ left moving wave

Now, we already encountered representing trig functions with complex notation $e^{i\theta} = \cos\theta + i\sin\theta$. and it really made things a lot easier (think: phasors) $\frac{d}{dx}e^x = e^x$ right? So we will use complex notation for waves as well.

<u>Complex Representation</u>

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}$$

this is a sinusoidal wave where the real part is our solution.

the tilde ~ reminds us of the complex parts.

so, $\tilde{A} \equiv Ae^{i\delta}$ (magnitude and phase shift)

So the physical wave is,

$$f(z,t) = \text{Re}[\tilde{f}(z,t)] = A\cos(kz-\omega t+\delta)$$ as before!

Note: If we ever need to sum up waves using <u>Fourier</u> we can,

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k)e^{i(kz-\omega t)}\,dk$$

produces any wave you want!

- This process sums <u>all</u> the waves with complex amplitudes and is fully general.
- The integral has + and − k's (as it must)

$$\tilde{f}(z,t=0) = \int_{-\infty}^{\infty} \tilde{A}(k)e^{ikz}\,dk$$

is in the standard Fourier form so we can find the $\tilde{A}(k)$'s,

$$\tilde{A}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(z,0)e^{-ikz}\,dz$$

3D Waves

the wave equation in 3D is,

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

In Cartesian this gives us,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

We won't check this but any function that has,

$$f(k_x x + k_y y + k_z z - \omega t + \delta) \quad \text{with } k_x^2 + k_y^2 + k_z^2 = \omega^2/v^2$$

can be a solution to this PDE. This solution is a wave travelling in the $\vec{k} = \langle k_x, k_y, k_z \rangle$ direction.

In 3D, our general sinusoidal solution is,

$$\tilde{f}(\vec{r},t) = \tilde{A} e^{i(\vec{k}\cdot\vec{r} - \omega t)} \qquad \text{where } \tilde{A} = A e^{i\delta}$$

$\vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$ tells you the direction of travel

$|k| = 2\pi/\lambda$ tells you the wavenumber

$|v| = \omega/|k|$ is the speed (called "phase velocity" b/c it's the rate of change of the phase in the exponential)

And as always

$$f_{physical}(\vec{r},t) = Re\left[\tilde{f}(\vec{r},t)\right] = A\cos(\vec{k}\cdot\vec{r} - \omega t + \delta)$$

This solution still has ∞ extent, a single definite ω, λ, and direction of travel $\rightarrow$ we can construct complex waves from these ideal solutions