Electromagnetic fields in Special Relativity

Consider the 4-force from our earlier work,

$$K^\mu = \frac{dp^\mu}{dt} = \left(\gamma \frac{\vec{F}\cdot\vec{v}}{c}, \gamma\vec{F}\right)$$

with $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$, we have

$$K^\mu = \left(\gamma q \frac{\vec{E}\cdot\vec{v}}{c}, \gamma q(\vec{E}+\vec{v}\times\vec{B})\right)$$

Because we know that K^μ is a 4 vector, we know how it transforms. We can use this to tell us how E & B transform; we won't derive this. One thing to notice is that E & B are not themselves 4 vectors so any tranformations will mix up E & B!

Minkowski showed we could rewrite K^μ in a helpful way,

$$K^\mu = q\eta_\nu F^{\mu\nu} \text{ where } F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

$F^{\mu\nu}$ is an antisymmetric tensor $F^{\mu\nu} = -F^{\nu\mu}$ (the field tensor)

This is an object that tells us how E & B transform, this tensor tranforms like this $F^{\mu\nu} = \Lambda^\mu_\gamma F^{\gamma\delta} \Lambda^\nu_\delta$

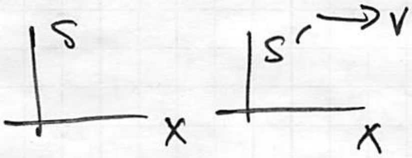
It's not ~~as~~ as simple as the Lorentz transformation, but it's close.

We will come back to it but suffice to say

$$K^0 = q\left[\eta_0 F^{00} + \eta_1 F^{01} + \eta_2 F^{02} + \eta_3 F^{03}\right]$$
$$= q\left[0 + \eta_x E_x/c + \eta_y E_y/c + \eta_z E_z/c\right]$$
$$= q\,\vec{\eta}\cdot\vec{E}/c = q(\gamma\vec{v}\cdot\vec{E}/c) \quad \text{as above.}$$

Just so you have them, here's the <u>results</u> for how E & B transform. Our buddy, Griffiths, derives them all by hand and there are a number of ways to get them including using the Minkowski formalism on the previous page.

Consider two frames as usual,



$$\left.\begin{aligned} t' &= \gamma(t - vx/c^2) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}\right\} \text{Here's the usual Lorentz transformation equations.}$$

Now,

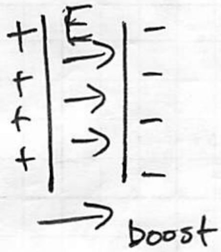
$$\left.\begin{aligned} E_x' &= E_x \\ E_y' &= \gamma(E_y - vB_z) \\ E_z' &= \gamma(E_z + vB_y) \\ B_x' &= B_x \\ B_y' &= \gamma\left(B_y + \tfrac{v}{c^2}E_z\right) \\ B_z' &= \gamma\left(B_z - \tfrac{v}{c^2}E_y\right) \end{aligned}\right\}$$

These are not the usual Lorentz eqn's, but they are perhaps quite simpler than you expected.

They arise directly from relativity.

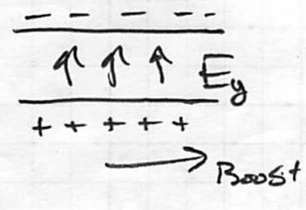
One way to see this is to consider two scenarios,

Scenario 1: capacitor boosted along its axis,



If you boost this configuration, the spacing contracts, but E is independent of x, so E_x does not transform.

Scenario 2:



If you boost this configuration Δx shrinks so that σ grows by γ! thus $E_y' = \gamma E_y$.

Griffiths does the rest of the formula using these kinds of conceptual explanations. Read It!

Sources of Fields (charges & currents)

Ok the sources of these E & B fields need to be unpacked to really understand how this all works.

Consider charges and currents in different frames

→ Charge is both conserved & Lorentz invariant (these are experimental facts!)

But charge densities (like above) and currents depend on derivatives and length scales, which make them frame dependent.

So we need to explore ρ & J in this context!

We will define a proper charge density,

$$\rho_0 \equiv \frac{Q}{V} \quad \text{in the charges rest frame (its invariant)}$$

Then $J^\mu \equiv \rho_0 \eta^\mu$ is a contravariant 4vector!

$$\eta^\mu = \gamma(c, \vec{u}) \quad \text{so that} \quad J^\mu \equiv (c\rho, \vec{J}) = \gamma(c\rho_0, \rho_0 \vec{u})$$

thus $\rho = \gamma \rho_0$ which makes sense the charge density is dilated by $\gamma \geq 1$ as the volume contracts Q/V $\xrightarrow{}$ invariant, $\longrightarrow$ contracts

Note that $\vec{J} = \rho \vec{u}$ as we said before!

Charge conservation

$\nabla \cdot \vec{J} + d\rho/dt = 0$ is still true in any frame

$$\frac{\partial J^0}{\partial x^0} = \frac{\partial (c\rho)}{\partial (ct)} = \frac{\partial \rho}{\partial t} \quad \text{cool!}$$

$$\frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3} = \nabla \cdot \vec{J} \quad \text{so that} \quad \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

can be rewritten as

$$\sum_\mu \frac{\partial J^\mu}{\partial x^\mu} = 0 \quad \text{or} \quad \partial_\mu J^\mu = 0 \quad \text{where } \partial_\mu \text{ is shorthand for } \frac{\partial}{\partial x^\mu}$$

Einstein double index

Not just elegant & compact, but $\partial_\mu J^\mu$ is manifestly invariant! Charge conservation is true in all frames!

with $J^\mu = \rho_0 \eta^\mu$ we find proper charge density is lorentz invariant so that charge is as well!

Let's go back to the $F^{\mu\nu}$ tensor we introduced.

I claim that,

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$$ is shorthand for Gauss' Law and Ampere's Law (with the Maxwell correction)

$\mu=0$ will give us — the rest will give us,

$\nabla \cdot \vec{E} = \rho/\epsilon_0$ and $\nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$

Let's see how one of these works.

But, note we have an elegant way of writing Maxwell's eqns in a "covariant form". They hold <u>in all frames</u> written this way.

<u>Example</u>: $\frac{\partial}{\partial x_\nu} F^{\mu\nu} = \mu_0 J^\mu$ Consider just $\mu=0$ (the L.H.S is summed over ν!)

$$\frac{\partial}{\partial x_\nu} F^{0\nu} = \mu_0 J^0 \Rightarrow$$ ($F^{0\nu}$ describes the first row of the tensor)

$$0 + \frac{\partial}{\partial x}\left(\frac{E_x}{c}\right) + \frac{\partial}{\partial y}\left(\frac{E_y}{c}\right) + \frac{\partial}{\partial z}\left(\frac{E_z}{c}\right) = \mu_0 c \rho$$

$$\Rightarrow \frac{1}{c}\nabla \cdot \vec{E} = \mu_0 c \rho \Rightarrow \nabla \cdot \vec{E} = \mu_0 c^2 \rho = \rho/\epsilon_0 !$$

You can derive the other one by considering the other μ's.

We can define one more antisymmetric tensor,

$$G^{\mu\nu} \equiv \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & +E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

This tensor transforms the same way as $F^{\mu\nu}$,

$$\overline{G}^{\mu\nu} = \Lambda^{\mu}_{\lambda} G^{\lambda\sigma} \Lambda^{\nu}_{\sigma}$$

It's called the <u>dual tensor</u>.

I claim that $\partial_\nu G^{\mu\nu} = 0$ is shorthand for these two Maxwell Eqns.

$\mu = 0$ gives $\nabla \cdot \vec{B} = 0$

$\mu \neq 0$ gives $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$

$\Big\}$ you can check it.

So we've written Maxwell's equations in a manifestly covariant way,

$$\left.\begin{aligned} \partial_\nu F^{\mu\nu} &= \mu_0 J^\mu \\ \partial_\nu G^{\mu\nu} &= 0 \end{aligned}\right]$$

These equations are frame independent, which are deep statements about ~~the~~ nature. They are true even when you boost or rotate!

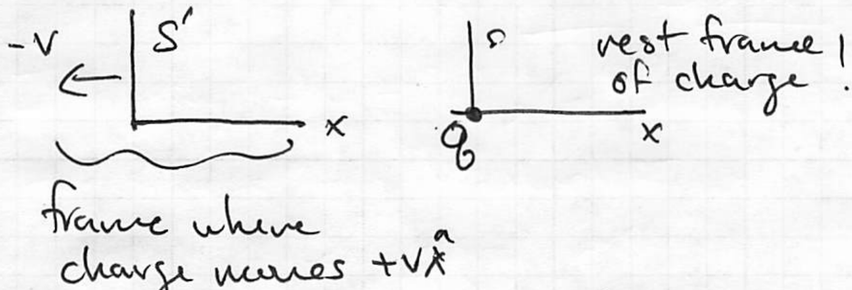
and $K^\mu = q\,\eta_\nu F^{\mu\nu}$ is the "force formula"

These 3 equations describe all of E&M, written compactly and consistently with relativity!

Final Example: The point charge (we end as we began)

Consider a pt. charge at rest. The field is the simple Coulomb $\vec{E}$ field, nothing special there.

So what's the EM field of a moving charge?



In S, $\begin{cases} \vec{E} = \dfrac{q}{4\pi\epsilon_0}\dfrac{\vec{r}}{r^3} = \dfrac{q}{4\pi\epsilon_0}\dfrac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}} \\ \vec{B} = 0 \end{cases}$

In S', we want to know E' & B' due to the moving charge. Note: ($v \to -v$ in this case)

$$\left.\begin{aligned} E_x' &= E_x \\ E_y' &= \gamma(E_y + vB_z) \\ E_z' &= \gamma(E_z - vB_y) \\ B_x' &= B_x \\ B_y' &= \gamma\left(B_y - \frac{v}{c^2}E_z\right) \\ B_z' &= \gamma\left(B_z + \frac{v}{c^2}E_y\right) \end{aligned}\right\}$$

Fields transform with $v \to -v$

also the Lorentz equ's $v \to -v$

$$\begin{cases} x = \gamma(x' - vt') \\ y = y' \\ z = z' \\ t = \gamma\left(t' - \frac{v}{c^2}x'\right) \end{cases}$$

So,
$$E_x' = E_x = \frac{q}{4\pi\epsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{q\gamma(x'-vt')}{4\pi\epsilon_0\left(\gamma^2(x'-vt')^2+y'^2+z'^2\right)^{3/2}}$$

<u>Note</u>: even though $E_x' = E_x$, there's still a coordinate transformation to the S' frame.

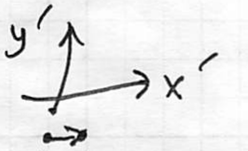
Next,
$$E_y' = \gamma E_y + 0 = \frac{q\gamma y'}{4\pi\epsilon_0\left(\gamma^2(x'-vt')^2+y'^2+z'^2\right)^{3/2}}$$

and similar for E_z'

So we can write these all together,
$$\vec{E}' = \frac{\gamma q}{4\pi\epsilon_0}\frac{\langle x'-vt', y', z'\rangle}{\left(\gamma^2(x'-vt')^2+y'^2+z'^2\right)^{3/2}}$$

At $t'=0$, when q passes the S origin,
$$\vec{E}' = \frac{\gamma q}{4\pi\epsilon_0}\frac{\langle x', y', z'\rangle}{(\gamma x'^2+y'^2+z'^2)^{3/2}}$$

Let's look at $z'=0$ in the plane of the page,



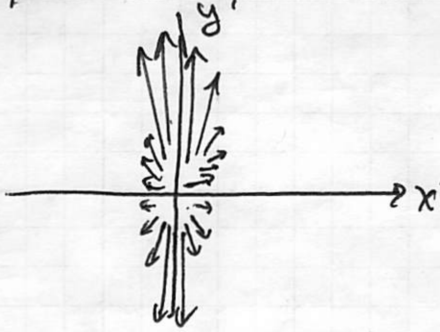
(with $z'=0$)

Along the $y'=0$ line, (the x-axis)
$$E'_{(x',0,0)} = \frac{\gamma q}{4\pi\epsilon_0}\frac{x'}{\gamma^3 x'^3} = \frac{q}{4\pi\epsilon_0}\frac{1}{\gamma^2 x'^2}$$

The E field is "γ^2 suppressed" along the direction of travel!

Along the y' axis ($x'=0$) $E'(0,y',0) = \frac{\gamma q}{4\pi\epsilon_0}\frac{y'}{y'^3} = \frac{q}{4\pi\epsilon_0}\frac{\gamma}{y'^2}$

The E field is "γ enhanced" $\perp$ to travel direction!



- E points away from the origin at $t'=0$
- At later t', it points away from wherever q is "now" in the S' frame.
- Enhanced in $\perp$ direction.

In S' $B \neq 0$! It's relatively easy to compute or to show with $\vec{B}' = -\frac{1}{c^2}\vec{v}\times\vec{E}'$.

The $\vec{B}$ field arises purely from the Lorentz transformation.

$$\left.\begin{aligned} B_x' &= 0 \\ B_y' &= -\frac{v}{c^2}\gamma E_z = \frac{v}{c^2}E_z' \\ B_z' &= +\frac{v}{c^2}\gamma E_y = -\frac{v}{c^2}E_y' \end{aligned}\right\}$$

The B field is a relativistic effect of an E-field viewed from a different perspective.