In our analysis of metals, we found that $|k|$ depends on ω. So the speed of the wave depends on its frequency and thus different "colors" of light will behave differently

If you build up a wave packet using Fourier's idea, the different frequency waves in the sum will travel at different speeds.

So a localized wave packet will spread out spatially. The different components will travel at different speeds, thus "dispersing" the wave packet over time. We call these media, "<u>dis</u><u>persive</u> <u>media</u>."

This physics yields <u>rainbows</u>! (different "n" for different ω)

- Conductors are dispersive, we found that $|k| \sim \sqrt{\omega}$, but they are also attenuating the field, so observing this in metals isn't a readily done (as in dielectrics)

- Dielectric materials are also dispersive. We treated them as linear with constant ε and thus n, which is independent of ω (when done this way). This turns out to be a bit of a oversimplification.

$\Rightarrow$ Let's return to dielectrics & model the interaction of the waves with molecules to <u>derive</u> the dispersion (we will find $\omega(k)$)

Our model will be crude & classical, but will capture the essential physics.

Let's first explore the wave packet,

Fourier tells us that we can construct a wave packet by summing the plane waves of different k's up,

$$f(x,t) = \int a(k)\, e^{i(kx - \omega t)}\, dk$$

$\Omega \rightarrow$ travelling wave $=$ sum of plane waves of different k's.

Aside: Without dispersion, $\frac{\omega}{k} = v$ is a constant for all k. With dispersion, $\omega = \omega(k)$, which need not be linear!

There's two takeaways that we will unpack in a moment,

(1) If you build a localized traveling packet, you need multiple k's, But if these are "concentrated" around some dominant central $k_0 = \omega_0/v_0$, then it turns out the wave packets "center" travels at a speed,

$$v_g = \left.\frac{d\omega}{dk}\right|_{k_0}$$

This is the "group velocity" versus the "phase velocity", $v_p = \omega/k$ of the plane waves (Each plane wave has a phase velocity)

(2) Relativity insists that $v_g < c$ for any physical waves. Information and energy travel at vgroup. (you can have $v_p > c$ in some cases, but you will never find $v_g > c$!)

For EM waves in matter, we want to know the "dispersion equation" $\omega(k)$ (or $k(\omega)$) so we can deduce $d\omega/dk$ and thus the speed of information travel.

Let's explore these ideas quantitatively,
Consider a pulse that is mostly "k_0" colored,

$$f(x) = \int_{-\infty}^{\infty} a(k)\, e^{ikx}\, dk \quad \text{with } a(k) = \sqrt{\frac{\sigma}{\pi}}\, e^{-\sigma(k-k_0)^2}$$

that is, the amplitudes are "Gaussian" with a peak at k_0.

then

$$f(x) = \sqrt{\frac{\sigma}{\pi}} \int_{-\infty}^{\infty} e^{-\sigma(k-k_0)^2} e^{ikx}\, dk$$

To simplify our work a bit, let's choose $k' = k - k_0$, $dk' = dk$

$$f(x) = \sqrt{\frac{\sigma}{\pi}} \int_{-\infty}^{\infty} e^{-\sigma(k')^2} e^{ik'x} e^{ik_0x}\, dk'$$

which we can transform in the following way,

$$-\sigma\left(k' - \frac{ix}{2\sigma}\right)^2 = -\sigma(k')^2 + \frac{ik'x}{\sigma}\sigma + \frac{\sigma x^2}{4\sigma^2}$$

$$= -\sigma(k')^2 + ik'x + \underbrace{\frac{x^2}{4\sigma}}_{\text{extra term so subtract off}}$$

that is,

$$f(x) = \sqrt{\frac{\sigma}{\pi}} \int_{-\infty}^{\infty} \underbrace{e^{-\sigma\left(k' - \frac{ix}{2\sigma}\right)^2} e^{-x^2/4\sigma}}_{\text{combine to give } e^{-\sigma(k')^2} e^{ik'x}} e^{ik_0x}\, dk'$$

$$f(x) = \sqrt{\frac{\sigma}{\pi}}\, e^{-x^2/4\sigma} e^{ik_0x} \int_{-\infty}^{\infty} e^{-\sigma(k' - ix/2\sigma)^2}\, dk'$$

let $k'' = k' - ix/2\sigma \qquad dk'' = dk'$

$$f(x) = \sqrt{\frac{\sigma}{\pi}}\, e^{-x^2/4\sigma} e^{ik_0x} \int_{-\infty}^{\infty} e^{-\sigma(k'')^2}\, dk''$$

$$\int_{-\infty}^{\infty} e^{-\sigma(k'')^2}\, dk'' = \sqrt{\frac{\pi}{\sigma}}$$ Gaussian Integral

$$f(x) = e^{-x^2/4\sigma} e^{ik_0 x}$$] this is localized in space (but with a phase that depends on primary color k_0)

OK what is this fourier sum in space when the wave is travelling?

$$f(x,t) = \int_{-\infty}^{\infty} a(k)\, e^{i(kx-\omega t)}\, dk$$ in free space, $\omega = v_p k$

$$= \int_{-\infty}^{\infty} a(k)\, e^{i(kx - v_p k t)}\, dk = \int_{-\infty}^{\infty} a(k)\, e^{ik(x - v_p t)}$$

$$= f(x - v_p t)$$ a pulse that moves with v_p! just another solution to the wave equation!

What about in media when $\omega = \omega(k)$?

$$\int_{-\infty}^{\infty} a(k)\, e^{i(kx-\omega t)}\, dk$$

when $\omega = \omega(k)$ we can't do the integral w/o knowing $\omega(k)$ or linearizing $\omega(k)$ around the peak (if that makes sense!)

Assume packet is peaked around k_0 then, ⇐

$$\omega(k) \simeq \omega(k_0) + (k-k_0)\left.\frac{d\omega}{dk}\right|_{k_0} + \ldots$$

Linearized $\omega(k)$ relationship around the peak k_0.

$$\omega(k) \cong v_p k_0 + (k-k_0) v_g + \ldots$$ We are choosing to center this around k_0 which is where $a(k)$ lives.

$$f(x,t) = \int_{-\infty}^{\infty} \sqrt{\frac{\sigma}{\pi}}\, e^{-\sigma(k-k_0)^2} e^{i(kx - v_p k_0 t - (k-k_0) v_g t)}\, dk$$

$$f(x,t) = \sqrt{\frac{\sigma}{\pi}} \int_{-\infty}^{\infty} e^{-\sigma(k-k_0)^2} e^{i(kx - v_p k_0 t - (k-k_0)v_g t)} dk$$

Try to clean up by writing $k = k - k_0 + k_0$

$$f(x,t) = \int_{-\infty}^{\infty} a(k-k_0+k_0)\, e^{i(k-k_0)(x-v_g t)}\, e^{-iv_p k_0 t}\, e^{ik_0 x}\, dk$$

$$= e^{-iv_p k_0 t} e^{ik_0 x} \int_{-\infty}^{\infty} a(k-k_0+k_0)\, e^{i(k-k_0)(x-v_g t)}\, dk \cancel{d(k-k_0)}$$

let $k' = k - k_0$ to make things a touch cleaner,

$$f(x,t) = e^{ik_0(x-v_p t)} \int_{-\infty}^{\infty} a(k'+k_0)\, e^{ik'(x-v_g t)}\, dk'$$

Note that,

$$\cancel{\int_{-\infty}^{\infty} a(k'+k_0)\, e^{ik'(x-v_g t)}\, dk'} \qquad \cancel{\text{let } k'' = k'+k_0}$$

$$\cancel{\int_{-\infty}^{\infty} a(k'')\, e^{ik''(x)}}$$

with $k'' = k' + k_0$

$$\int_{-\infty}^{\infty} a(k'+k_0)\, e^{ik'x}\, dk' = \int_{-\infty}^{\infty} a(k'')\, e^{ik''x}\, dk''\, e^{-ik_0 x}$$

$$= e^{-ik_0 x} \underbrace{\int_{-\infty}^{\infty} a(k'')\, e^{ik''x}\, dk''}_{\text{integral of the "stationary" pulse, } f(x)} = e^{-ik_0 x} f(x)$$

$$f(x,t) = e^{ik_0(x-v_p t)} \int_{-\infty}^{\infty} a(k'+k_0)\, e^{ik'(x-v_g t)}\, dk'$$

will give us,

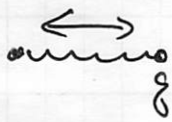
$$f(x,t) = e^{ik_0(x-v_p t)} \left[e^{-ik_0(x-v_g t)} f(x-v_g t) \right]$$

$$f(x,t) = \underbrace{e^{ik_0(v_g - v_p)t}}_{\text{time dependent phase}} f(x - v_g t) \quad \text{travels at } v_g!$$

Let's try to understand where this frequency dependence might come from. We're going to build a simple classical model that helps explain this, but know that the effects are really a quantum phenomenon!

Atoms are charges on springs

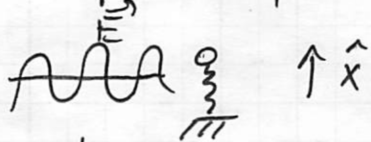
→ our model will assume that the force experienced by electrons are "spring-like" but there's some drag we will model as well that is connected to the radiation of energy due to the charge accelerating.



In an electric field the force on the charge will be,

$$\vec{F}_{net} = q\vec{E} - k_s\vec{x} - C\vec{v}$$

We will assume a 1D model aligned w/ the polarization of the field.



$k_s = m\omega_0^2$ (ω_0 is the resonant frequency of the spring)

$C \equiv m\gamma$ is some internal friction (due to radiation)

Let's drive the charge with a harmonic field,

$$\vec{E} = E_0\hat{x}e^{i(kz-\omega t)}$$

Focusing on any single atom (at $z=0$) we have a simple model,

$$m\ddot{x} = \underbrace{qE_0e^{-i\omega t}}_{\text{driver}} - \underbrace{m\omega_0^2 x}_{\text{spring}} - \underbrace{m\gamma\dot{x}}_{\text{damping}}$$

$x(t)$ displacement of electron

$$m\ddot{x} = qE_0 e^{-i\omega t} - m\omega_0^2 x - m\gamma\dot{x}$$

is a familar ODE. We try the ansatz,

$$x(t) = x_0 e^{-i\omega t}$$

$$-m\omega^2 x_0 = qE_0 - m\omega_0^2 x_0 - m\gamma(-i\omega)x_0 \quad \left(\text{after we cancel the } e^{-i\omega t}\text{'s}\right)$$

this equation is solved if x_0 takes a specific value,

$$x_0 = \frac{qE_0}{m}\frac{1}{(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

with this x_0 the dipole moment of any atom oscillates,

$$p(t) = qx(t) = q\underbrace{x_0 e^{-i\omega t}}_{\text{just the solution to the position ODE.}}$$

So our electric field polarizes the atoms,

$$\vec{p} = \frac{q^2E/m}{\omega_0^2-\omega^2-i\gamma\omega}e^{-i\omega t}\hat{x} = \frac{q^2/m}{\omega_0^2-\omega^2-i\gamma\omega}\vec{E}(t)$$

So $\vec{p} \propto \vec{E}$ but the constant of proportionality is now complex! $\vec{p}$ is out of phase w/ $\vec{E}$.

$\vec{p}$ "lags" $\vec{E}$

<u>Bulk Polarization</u>

As we have done in the past, all the little dipoles add up to give the ~~p~~ polarization

$\vec{P} = N\vec{p}$ where N is the # molecules/volume.

If each molecule has "f_j" electrons, each with its own resonant frequency, "ω_j", and damping "γ_j", then we really need to construct the sum,

$$\vec{P} = \frac{Nq^2}{m} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\gamma_j \omega)} \tilde{E}$$

Note: we assume a dilute gas, because only E_{ext} is polarizing.

In dense materials, $\vec{P}$ creates its own field that causes additional E fields that polarize.

So we have,

$$\underbrace{\vec{D} = \varepsilon \vec{E}}_{\text{linear}} = \underbrace{\varepsilon_0 \vec{E} + \vec{P}}_{\text{always}} = \varepsilon_0 \vec{E}\left(1 + \frac{Nq^2}{m\varepsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\gamma_j\omega)}\right)$$

So we have a new formula for $\varepsilon = \varepsilon(\omega)$, which is both complex & depends on frequency.

Let's go back to the wave equation in a linear dielec.

$$\nabla^2 \vec{E} = \varepsilon\mu \, \partial^2 \vec{E}/\partial t^2$$

[still true but now $\varepsilon = \tilde{\varepsilon}(\omega) \rightarrow$ anything we've done still holds!]

OLD solution still works,

$$\vec{E} = \tilde{\vec{E}} e^{i(\tilde{k}z - \omega t)} \quad \text{but with} \quad \frac{\tilde{k}}{\omega} = \sqrt{\tilde{\varepsilon}\mu_0}$$

So $\tilde{k} = \sqrt{\tilde{\varepsilon}(\omega)\,\omega^2 \mu_0}$ is complex (which means losses) & depends nonlinearly on ω (dispersive)

We learned in the last section how to work w/ complex $\tilde{k}$,

$$\tilde{k} = k_R + i k_{IM}, \quad \text{so} \quad \vec{\tilde{E}} = \vec{\tilde{E}}_0 e^{-k_{Im} z} e^{i(k_R z - \omega t)}$$

the damping arises physically from "friction" in our model and is not what we learned from conductors. They come from model parameters $(m_e, q, \gamma, \omega_0^2, \ldots)$.

$$\tilde{\varepsilon} = \varepsilon_0 \left(1 + \frac{Nq^2}{m\varepsilon_0} \underbrace{\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega}} \right)$$

In the dilute gas approx, this should be small. so that,

$$\frac{\tilde{k}}{\omega} = \sqrt{\mu_0 \tilde{\varepsilon}} = \sqrt{\mu_0 \varepsilon_0}\sqrt{1 + \text{small}} \simeq \sqrt{\mu_0 \varepsilon_0}\left(1 + \tfrac{1}{2}\text{small}\right)$$

with $\tilde{k} = k_R + i k_{Im}$ so we need to deal with the real and imaginary parts of the sum,

$$\mathrm{Re}\left(\frac{1}{a - bi}\right) = \mathrm{Re}\left(\frac{1}{a-bi}\,\frac{a+bi}{a+bi}\right) = \frac{a}{a^2+b^2}$$

$$\mathrm{Im}\left(\frac{1}{a-bi}\right) = \frac{b}{a^2+b^2}$$

this lets us just read off,

$$k_R = \frac{\omega}{c}\left(1 + \frac{Nq^2}{2m\varepsilon_0} \sum_j \frac{f_j(\omega_j^2 - \omega^2)}{\left[(\omega_j^2 - \omega^2)^2 + (\gamma_j \omega)^2\right]} \right)$$

$$k_{Im} = \frac{\omega}{c} \frac{Nq^2}{2m\varepsilon_0} \sum_j \frac{f_j \gamma_j \omega}{\left[(\omega_j^2 - \omega^2)^2 + (\gamma_j \omega)^2\right]}$$