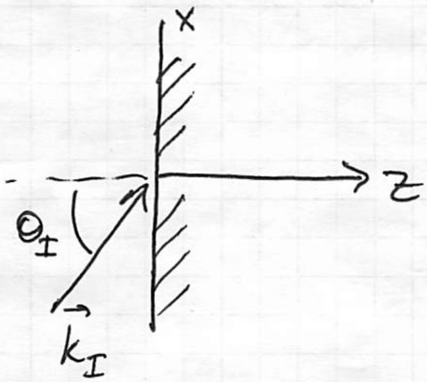
So we've developed a description for what happens when a wave strikes the boundary between two media, normally incident to the boundary.
What happens when it strikes at a oblique angle?

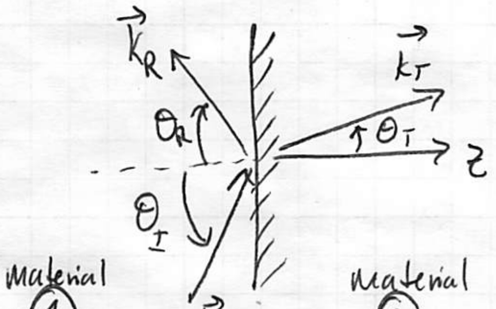
Consider a wave entering at some angle,



there is a plane defined by $\vec{k}_I$ and z, which is called the "plane of incidence". It's the plane of the page here.

We define the x-axis as shown so that the "plane of incidence" is the xz plane.

<u>Assumptions</u>: Again we assume monochromatic plane waves. And thus, our Boundary conditions will tells us everything we need to know.



$$\tilde{\vec{E}}_I = \tilde{\vec{E}}_{0I}\, e^{i(\vec{k}_I\cdot\vec{r}-\omega_I t)} \qquad \tilde{\vec{B}}_I = \frac{\hat{k}_I \times \tilde{\vec{E}}_I}{v_1}$$

$$\tilde{\vec{E}}_R = \tilde{\vec{E}}_{0R}\, e^{i(\vec{k}_R\cdot\vec{r}-\omega_R t)} \qquad \tilde{\vec{B}}_R = \frac{\hat{k}_R \times \tilde{\vec{E}}_R}{v_1}$$

$$\tilde{\vec{E}}_T = \tilde{\vec{E}}_{0T}\, e^{i(\vec{k}_T\cdot\vec{r}-\omega_T t)} \qquad \tilde{\vec{B}}_T = \frac{\hat{k}_T \times \tilde{\vec{E}}_T}{v_2}$$

As before, we cannot get that $\vec{E}$ & $\vec{B}$ are continuous without all the ω's being the same (i.e., $\omega = \omega_I = \omega_R = \omega_T$)

$$\tilde{\vec{E}}_{I} = \tilde{\vec{E}}_{0I}\, e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}$$

$$\tilde{\vec{E}}_R = \tilde{\vec{E}}_{0R}\, e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}$$

$$\tilde{\vec{E}}_T = \tilde{\vec{E}}_{0T}\, e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}$$

All of our Boundary Conditions result in the same expression

$$(Blah_I)e^{i(\vec{k}_I\cdot\vec{r}-\omega t)} + (Blah_R)e^{i(\vec{k}_R\cdot\vec{r}-\omega t)} = (Blah_T)e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}$$

*We can examine these to see in what plane(s) the waves propagate.

The blah's are complex vectors but have no $\vec{r}$ or t dependence.

When we match boundary conditions in the $z=0$ plane, the vector $\vec{r}$ will be, $\vec{r}=\langle x,y,0\rangle$. Each term has an $e^{-i\omega t}$ that cancels out so you are left with,

$$(Blah_I)e^{i(\vec{k}_I\cdot\vec{r})} + (Blah_R)e^{i(\vec{k}_R\cdot\vec{r})} = (Blah_T)e^{i(\vec{k}_T\cdot\vec{r})}$$

By the same logic as before the exponential terms must all be the same in the $z=0$ plane,

$\vec{k}_I\cdot\vec{r} = \vec{k}_R\cdot\vec{r} = \vec{k}_T\cdot\vec{r}$ for <u>all</u> $\vec{r}$ in $z=0$ plane.

$\vec{k}_I = k_{Ix}\hat{x} + k_{Iz}\hat{z}$ is given (it's the incident wave so there's no k_{Iy}.)

We can argue that, $(\vec{k}_I - \vec{k}_R)\cdot\vec{r} = 0$ for <u>any</u> $\vec{r}=\langle x,y,0\rangle$

As $\vec{r}$ is some arbitrary vector we find that,

$$k_{Ix}x - (k_{Rx}x + k_{Ry}y) = 0 \quad \text{for all } x \text{ and } y$$

so $k_{Ry}=0$ (it must be for the above to hold for any y.)

↳ the reflected wave is also in the page!

also $k_{Ix} = k_{Rx}$ (we will come back to this).

Similarly $\vec{k}_I \cdot \vec{r} = \vec{k}_T \cdot \vec{r}$ tells us that

$$k_{Ix}\, x - (k_{Tx}\, x + k_{Ty}\, y) = 0$$

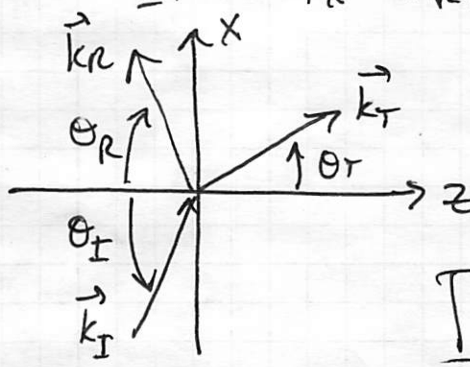
so here too $k_{Ty} = 0$ transmitted wave is "in the page"

and $k_{Ix} = k_{Tx}$ (come back to this in a moment)

[Reminder: $k = \frac{\omega}{v}$ always and this will be useful]

Let's look back at our picture with the knowlege

that $k_{Ix} = k_{Tx} = k_{Rx}$



If $k_{Ix} = k_{Rx}$, then

$k_I \sin\theta_I = k_R \sin\theta_R$

but $k_I = k_R = \omega/v_1$ so that,

$$\boxed{\sin\theta_I = \sin\theta_R}$$

the angle of incidence = the angle of reflection

This comes purely from Maxwell's Equations!?

If $k_{Ix} = k_{Tx}$ then $k_I \sin\theta_I = k_T \sin\theta_T$.

In this case $k_T/k_I = \dfrac{\omega/v_2}{\omega/v_1} = \dfrac{v_1}{v_2} = \dfrac{n_2}{n_1}$ so that,

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{k_T}{k_I} = \frac{n_2}{n_1} \Rightarrow \boxed{n_1 \sin\theta_I = n_2 \sin\theta_T}$$

Snell's Law of refraction comes from

Maxwell's Equations!?

- this is general <u>law</u> for waves at a boundary
- We haven't really use ME's yet.

Snell's comes from assuming linear, homogeneous media.

The work we have done so far tells us that when we begin to apply Maxwell's equations the exponential terms will cancel out at the boundary as,

$$\vec{k}_I \cdot \vec{r} = \vec{k}_T \cdot \vec{r} = \vec{k}_R \cdot \vec{r} \quad \text{and} \quad \omega_R = \omega_I = \omega_T$$
for all $\underline{\vec{r}}$ — for all $\underline{t}$

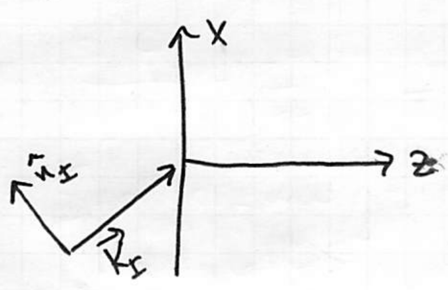
<u>We can completely neglect them!</u>

- You might be happy that we found the law of reflection and Snell's law of refraction, but we can actually dive deeper into this and find the transmitted & reflected amplitudes and intensities!
- To do this, we will need to specify a polarization of the incident wave,

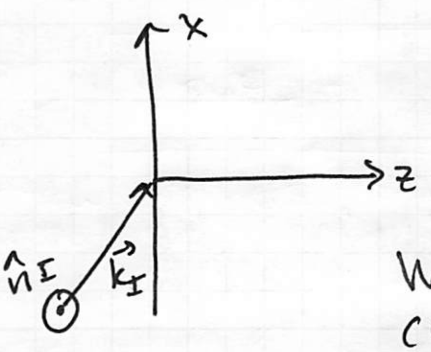
$$\vec{E}_I \sim \hat{n}_I \quad \text{(it will matter!)}$$

There are two cases here:

1) $\hat{n}_I$ lies in our plane of incidence ("the page")
2) $\hat{n}_I$ is $\perp$ to our plane of incidence



Case 1



Any other case is a linear combination!

We'll work out case 1 (case 2 is part of your hw)

Case 1) $\hat{n}_I$ has no y component so that,

$$\hat{k}_I = \sin\theta_I\,\hat{x} + \cos\theta\,\hat{z}$$
$$\hat{n}_I = \cos\theta_I\,\hat{x} - \sin\theta\,\hat{z}$$

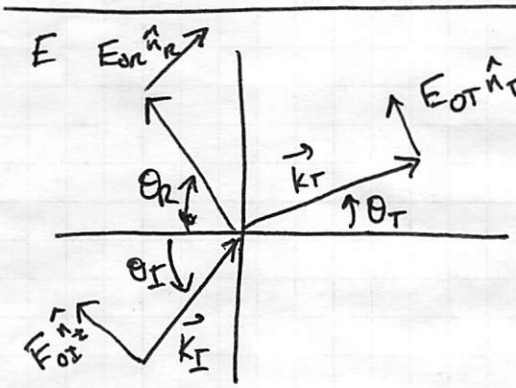
} Convince yourself that these make sense!

[Quick check $\hat{k}_I \cdot \hat{n}_I = \sin\theta_I\cos\theta_I - \cos\theta_I\sin\theta_I = 0$ ✓]

Reminder about Boundary Conditions

$$\varepsilon_1 E_1^\perp = \varepsilon_2 E_2^\perp$$
$$B_1^\perp = B_2^\perp$$
$$E_1^{\parallel} = E_2^{\parallel}$$
$$B_1^{\parallel} = B_2^{\parallel} \quad (\text{where } \mu_1 \simeq \mu_2 \simeq \mu_0)$$

- All of these hold at $z=0$
- $\perp$ is $\perp$ to boundary
- for case 1 $B_1^\perp = B_2^\perp$ tells us nothing b/c no B component $\perp$ to boundary.
- Last eqn is redundant (as it turns out)



Here's our picture now with the direction of the electric field labelled.

Using $\varepsilon_1 E_1^\perp = \varepsilon_2 E_2^\perp$,

$$\varepsilon_1\left(\tilde{E}_{I0}(\hat{n}_I)_z + \tilde{E}_{0R}(\hat{n}_R)_z\right) = \varepsilon_2 \tilde{E}_{0T}(\hat{n}_T)_z$$

(we cancelled $e^{i(\dots)}$ as we said earlier)

the z-components give,

$$\varepsilon_1\left(\tilde{E}_{I0}(-\sin\theta_I) + \tilde{E}_{0R}(\sin\theta_R)\right) = \varepsilon_2 \tilde{E}_{T0}(-\sin\theta_T)$$

Using Snell's Law gives us $\sin\theta_I = \sin\theta_R$ and

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I$$

So that,

$$\varepsilon_1\left(\tilde{E}_{0I}(-1)+\tilde{E}_{0R}\right)=\varepsilon_2\tilde{E}_{0T}(-1)\left(\frac{n_1}{n_2}\right)$$

or

$$\varepsilon_1\left(\tilde{E}_{0R}-\tilde{E}_{0I}\right)=-\varepsilon_2\frac{n_1}{n_2}\tilde{E}_{0T}$$ from B.C. #1.

We now use the B.C., $E_1^{\parallel}=E_2^{\parallel}$ ($\parallel$ to boundary, x components),

$$\tilde{E}_{0I}\cos\theta_I+\tilde{E}_{0R}\cos\theta_R=\tilde{E}_{0T}\cos\theta_T$$ ($B^{\parallel}$ gives no new info)

We have 2 eqn's and 2 unknowns ($\tilde{E}_{0T}$ and $\tilde{E}_{0R}$),

$$\tilde{E}_{0I}-\tilde{E}_{0R}=\frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}\tilde{E}_{0T}=\beta\tilde{E}_{0T}$$ (same β as before!)

$$\tilde{E}_{0I}+\tilde{E}_{0R}=\tilde{E}_{0T}\frac{\cos\theta_T}{\cos\theta_I}=\alpha\cdot\tilde{E}_{0T}$$ (α is a constant given θ_I Snell's gives us θ_R so α is determined)

These equations produce!

$$\tilde{E}_{0T}=\frac{2}{(\alpha+\beta)}\tilde{E}_{0I} \qquad \tilde{E}_{0R}=\left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0I}$$

Again these are the Fresnel equations when $\theta_R=\theta_T=\theta_I=0^\circ$ $\cos\to 1$ so $\alpha\to 1$!

Case 1) $\hat{n}_I$ has no y component so that,

$$\hat{k}_I = \sin\theta_I\,\hat{x} + \cos\theta\,\hat{z}$$
$$\hat{n}_I = \cos\theta_I\,\hat{x} - \sin\theta\,\hat{z}$$

} Convince yourself that these make sense!

[Quick check $\hat{k}_I \cdot \hat{n}_I = \sin\theta_I\cos\theta_I - \cos\theta_I\sin\theta_I = 0$ ✓]

Reminder about Boundary Conditions

$$\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp}$$
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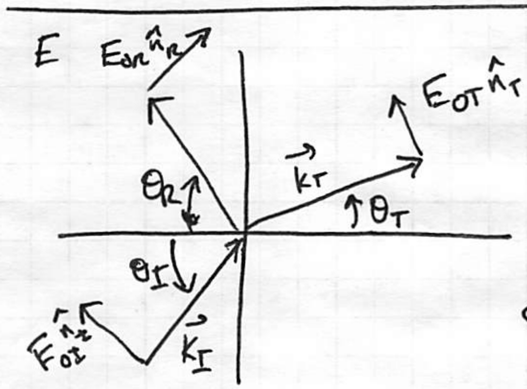


Here's our picture now with the direction of the electric field labelled. $\Leftarrow$

Using $\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp}$,

$$\varepsilon_1\left(\tilde{E}_{I0}(\hat{n}_I)_z + \tilde{E}_{0R}(\hat{n}_R)_z\right) = \varepsilon_2 \tilde{E}_{0T}(\hat{n}_T)_z$$

(we cancelled $e^{i(\dots)}$ as we said earlier)

the z-components give,

$$\varepsilon_1\left(\tilde{E}_{I0}(-\sin\theta_I) + \tilde{E}_{0R}(\sin\theta_R)\right) = \varepsilon_2 \tilde{E}_{T0}(-\sin\theta_T)$$

Using Snell's Law gives us $\sin\theta_I = \sin\theta_R$ and

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I$$

A few observations:

1) $\tilde{E}_{OT}$ is always in phase w/ $\tilde{E}_{OI}$. The reflected wave can pickup a minus sign when $\alpha < \beta$. But no complex phases are introduced.

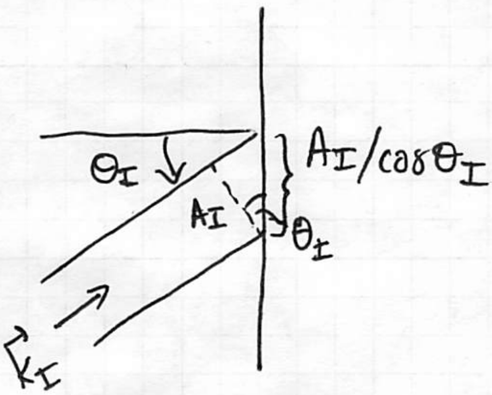
2) The B-fields all come from $\hat{k} \times \vec{E}/v$.

3) Unlike the normal incidence Example, $\tilde{E}_{OR}$ and $\tilde{E}_{OT}$ do depend on θ_I, they are not simply determined by n_1 & n_2.

4) If $\alpha = \beta$, then $\tilde{E}_{OR} = 0$. Whoa! that's interesting. This is a special angle (Brewster's Angle) defined by $\alpha = \beta \Rightarrow \dfrac{\cos\theta_I}{\cos\theta_T} = \dfrac{n_1}{n_2}$

At this angle all the light transmits, but this only for case 1. when the polarization is in the plane.

Intensity of Reflected and Transmitted Light



Geometrically the magnitude of the Poynting vector will be given by,

$$|\vec{S}| = \frac{\text{Power in}}{\text{Area}, \perp} \quad \text{so the power is,}$$

$$\text{Power} = |\vec{S}| \cdot \text{Area}_\perp = |\vec{S}| A_I$$ But the intensity is,

$$\text{Intensity} = \frac{\text{Power}}{\text{wall Area}} = \frac{|S|\, A_I}{(A_\perp / \cos\theta)} = |S| \cos\theta\, !$$

More formally, $I = \langle \vec{S} \rangle \cdot \hat{z} \Rightarrow |S| \cos\theta$

Thus,
$$I_I = \langle S_I \rangle \cos\theta_I$$
$$I_R = \langle S_R \rangle \cos\theta_R$$
$$I_T = \langle S_T \rangle \cos\theta_T$$

So that, $$R \equiv \frac{I_R}{I_I} = \frac{E_{0R}^2}{E_{0I}^2} = \frac{(\alpha - \beta)^2}{(\alpha + \beta)^2} \quad \left(\begin{array}{l} \text{with } \cos\theta_I = \cos\theta_R \\ \text{and} \\ I = \tfrac{1}{2} \varepsilon v E^2 \end{array} \right)$$

T is a bit more complicated,

$$T \equiv \frac{I_T}{I_I} = \frac{n_2}{n_1} \frac{E_{0T}^2}{E_{0I}^2} \frac{\cos\theta_T}{\cos\theta_I} = \frac{4\alpha\beta}{(\alpha + \beta)^2}$$

↑ this is α

Once again, $R + T = 1$. ("conservation of energy")

Check out the plots in Griffith's !

Finally, if $n_2 < n_1$ (e.g. going from water to air), there is a critical angle θ_{crit}.

$\sin\theta_{crit} = n_2/n_1$ where $\theta_T = 90°$

When $\theta_I > \theta_{crit}$, you cannot transmit <u>any</u> light. Total internal reflection.

→ Many Applications → optical fibers transmitting light!

* What happens if we try to use our formal results for $\theta_I > \theta_{crit}$? $\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I > 1$!

$$\tilde{\vec{E}}_T = \tilde{\vec{E}}_{0T}\, e^{i(\vec{k}_2\cdot\vec{r} - \omega t)} = \tilde{\vec{E}}_{0T}\, e^{i(k_{2x}x + k_{2y}y + k_{2z}z - \omega t)}$$

Here $k_2 = \frac{\omega}{c} n_2$ formally, $k_{2x} = k_2\sin\theta_T = \frac{\omega n_2}{c}\sin\theta_T = \frac{\omega n_1}{c}\sin\theta_I$

$k_{2y} = 0$

$k_{2z} = k_2\cos\theta_T = \frac{\omega n_2}{c}\underbrace{\sqrt{1-\sin^2\theta_T}}$

So, $\tilde{\vec{E}}_T = \tilde{\vec{E}}_{0T}\underbrace{e^{-\frac{\omega n_2}{c}\sqrt{\sin^2\theta - 1}\, z}} e^{i\omega\left(\frac{n_1\sin\theta_I}{c}x - t\right)}$ <u>imaginary</u>

exponentially dies in z direction!

No energy flow in +z direction, but $E \neq 0$

"Evanescent wave" classical effect that is reminiscent of quantum tunnelling!