Displacement Currents and Magnetic Fields

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(Received 31 October 1962)

The methods of calculating fields due to quasi-steady currents in closed and unclosed circuits are reviewed. It is emphasized that it is sufficient to apply the Biot–Savart law to all the moving charges and to ignore the vacuum displacement current. Attention is drawn to a basic error in the widespread practice of treating a circuit containing a capacitor as though the fringing fields could be ignored and the displacement current replaced by conduction currents confined to the gap between the plates.

1. INTRODUCTION

Surprisingly many textbooks of physics contain misleading or erroneous statements about the magnetic effect of electric displacement currents. To focus attention on the type of misconception that is involved, let us consider the following problem: Suppose that we charge two parallel conducting plates so that they carry equal charges of opposite sign; between them there is an electric field. Now imagine that we turn on a strong x-ray source and irradiate the air between the plates. The air in the gap becomes ionized and the ions move, carrying electric charges from one plate to the other, thereby discharging the plates. What is the magnetic field at a point, say P, outside the plates? (Fig. 1)

Certainly the movement of the ions produces a magnetic field at P. It is sometimes argued that, in addition, as the plates are discharged, the changing electric field between the plates, that is, the displacement current, produces a magnetic field at P which is equal and opposite to that produced by the movement of the ions; it is thus predicted that the resultant magnetic field is zero. This is incorrect. The movement of the ions produces a magnetic field at P, and that *is* the field. It is not canceled or diminished by anything else.[1] Planck expressed this clearly many years ago when he wrote,[2] ". . . even in the case of unclosed currents the magnetic intensity of the field is calculated from the vector-potential of the conduction currents without regard to the displacement currents. . . ." (Planck is here dealing with situations for which ϵ is the same everywhere.)

We will return to this "leaky capacitor" problem later, but first let us examine more generally the role of the displacement current in such problems.

2. CALCULATING THE MAGNETIC FIELD

There are several equivalent methods by which the magnetic field can be calculated. For quasi-steady conditions, that is, ignoring radiation effects, **B** can be calculated, using the Biot–Savart law, from a knowledge of the *real currents alone*:

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')d\tau'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{1}$$

This is one method.

A second method is to use the Maxwell equation

$$\text{curl }\mathbf{B}=\mu_0(\mathbf{J}+\dot{\mathbf{D}}). \tag{2}$$

This equation can be obtained by taking the curl of Eq. (1), provided that quasi-steady conditions are assumed.[3] If Eq. (2) is integrated over a Stokes surface bounded by some closed contour, we have the familiar result

$$\oint\mathbf{B}\cdot d\mathbf{l}=\mu_0\int_S(\mathbf{J}+\dot{\mathbf{D}})\cdot d\mathbf{S}. \tag{3}$$

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‡ The work of the Science Teaching Center is supported by a grant from the National Science Foundation.

[1] Any conduction currents which flow in the plates, of course, also contribute to the magnetic field.

[2] Max Planck, translated by H. L. Brose, *Introduction to Theoretical Physics* (MacMillan and Company, Ltd., London, 1932), Vol. III, p. 197; see also W. T. Scott, *The Physics of Electricity and Magnetism* (John Wiley & Sons, Inc., New York, 1959), p. 304.

[3] See, for example, Arthur Bierman, Am. J. Phys. **29**, 355 (1961); J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1962), Secs. 5.3, 6.3.

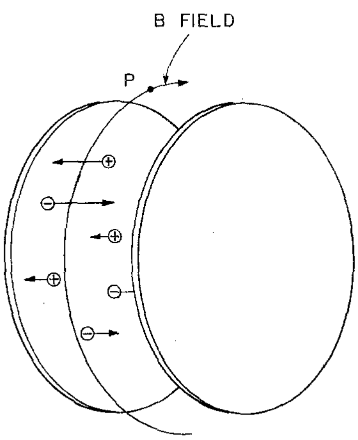

FIG. 1. Magnetic field surrounding the "leaky capacitor."

This integrated form then allows one to make the statement that the line integral of **B** around a closed loop is equal to the total flux of real current plus displacement current through the loop. We have here what we may call the generalized form of Ampère's circuital theorem. In this way of stating the result, the inclusion of the displacment current is essential for obtaining correct answers.

It may be instructive to illustrate the equivalence of these two methods with the following specific and basic example: Consider a current I flowing over an infinitesimal line segment ds (Fig. 2) with accumulation of charges $-q$ and $+q$ at the ends of the element.

Let us calculate, by means of the circuital theorem, the magnetic field at a point P, whose coordinates are (r,ϑ) with respect to one end of the element. Given the obvious symmetry of the problem, we choose as the boundary line l of our integration a circle through P, of radius $r \sin\vartheta$, with its center on the axis defined by ds. The surface of integration on the right-hand side of Eq. (3) can be any surface bounded by the contour.

Suppose, first, that we take a surface S that does not intersect the current element. Then the **J** term in Eq. (3) vanishes, and, ignoring retardation effects, we have,

$$\int_S \dot{\mathbf{D}}_{(+q)} \cdot d\mathbf{S} = \tfrac{1}{2}\dot{q}(1-\cos\vartheta)$$

$$\int_S \dot{\mathbf{D}}_{(-q)} \cdot d\mathbf{S} = -\tfrac{1}{2}\dot{q}[1-\cos(\vartheta+d\vartheta)].$$

Combining these, and using $d\vartheta = -ds \sin\vartheta/r$, we have

$$\int_S \dot{\mathbf{D}} \cdot d\mathbf{S} = \tfrac{1}{2}\dot{q} \sin^2\vartheta ds/r.$$

The circuital theorem then gives us

$$2\pi r \sin\vartheta B_l = (\mu_0 \dot{q}/2)(\sin^2\vartheta ds/r),$$

that is

$$B_l = (\mu_0/4\pi)(I \sin\vartheta ds/r^2). \quad (4)$$

This is at once recognized as the result obtained by applying the Biot–Savart law to the real current flowing in the segment ds.[4]

It can be easily verified that if we had used a surface S', intersecting the current element, the **J** and $\dot{\mathbf{D}}$ would have combined to give the same result.

The above example of the equivalence of Eqs. (1) and (3), though simple, is important, because any electrical circuit, whether it be closed or open, can be represented as a combination of such current elements. And the currents in successive elements can be different, thus allowing for the accumulation of charge at particular places in the circuit, as, for example, on the plates of a capacitor.

3. CIRCUITS WITH CAPACITORS

We consider now a parallel-plate capacitor being charged by a current I [Fig. 3(a)]. Let the capacitor have circular plates of radius R and separation d, and suppose that we wish to find the magnetic field at a point P, lying just outside the plates on the mid-plane between

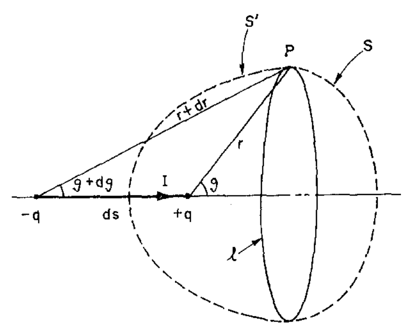

FIG. 2. Magnetic field due to a current element.

[4] For excellent further discussion, see E. G. Cullwick, *The Fundamentals of Electro-Magnetism* (Cambridge University Press, New York, 1949), pp. 149 ff.

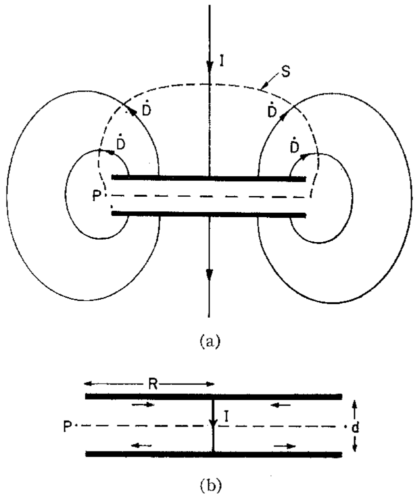

FIG. 3. (a) Capacitor with fringing fields, during charging; (b) current flow in internally shorted capacitor.

them. The first feature to be noted is that the field at P is certainly less than would be the case if the real current I were continued between the plates. For suppose we draw a Stokes surface S as shown, having as its boundary a circular loop through P with its center on the axis of symmetry. This surface will be threaded in one direction by the real current I, and in the opposite direction by lines of $\dot{\mathbf{D}}$. If the capacitor plates were shorted, without changing the size of I, the potential difference between them would become constant, and the lines of $\dot{\mathbf{D}}$ would disappear. Thus the net flux of conduction plus displacement currents through the surface S, and hence the field $\mathbf{B}$, is less in the first case than in the second. But how much less? To say (as many textbooks do) that the difference is trivial, being due merely to the small fringing fields, covers up a vital feature: the difference corresponds precisely to the missing part of the real-current circuit.

A closely related situation is the internal discharge of a capacitor, and we believe that this example is of particular value in exposing the basis of some of the misconceptions regarding displacement current. In Fig. 3(b) we show our capacitor being discharged by a shorting wire between the plates. The general pattern of current flow is shown by the arrows. Again we consider a circular path through all points such as P, distant R (approximately) from the axis of symmetry, but this time we take our Stokes surface to be the plane of the circle. The circuital law tells us that

$$2\pi R B_l = \mu_0\left(I + \int_S \dot{\mathbf{D}}\cdot d\mathbf{S}\right),$$

where I is the total current flowing in the shorting wire. Hence

$$B_l = (\mu_0 I/2\pi R) + (\mu_0/2\pi R)\int_S \dot{\mathbf{D}}\cdot d\mathbf{S}. \qquad (5)$$

We thus have the magnetic field formally expressed as the algebraic sum of two terms: the field of an infinitely long wire carrying the current I, and the field due to an oppositely directed flow of displacement current. The usual elementary discussion of this result consists in saying that, *if fringing effects are ignored*, the two terms on the right-hand side of Eq. (5) are equal and opposite, and hence the external magnetic field is zero. This is a mistake in principle. It requires that the electric field be entirely confined to the region between the plates; but this in turn requires reducing the gap between the plates to zero, thus eliminating the capacitor altogether. If we analyze the problem from the standpoint of the Biot–Savart law, then, as Fig. 3(b) shows, the real currents in the capacitor plates and the shorting wire will cooperate to give a nonvanishing field at P, and its value will be given approximately by

$$B_l \approx (\mu_0/4\pi)(Id/R^2). \qquad (6)$$

It is no surprise to discover that the actual field B_l, as given by Eq. (6), is of the order of d/R times the first term on the right-hand side of Eq. (5).

4. THE LEAKY CAPACITOR

We have discussed the two most familiar methods by which one can calculate the magnetic field in the neighborhood of a circuit containing a capacitor. The relationship between them can be further illuminated, and the importance of the fringing field emphasized, by the following considerations, which we will apply to the leaky capacitor problem with which we began this paper.

Equations (2) and (3) show clearly that the displacement current density $\dot{\mathbf{D}}$ is equivalent to

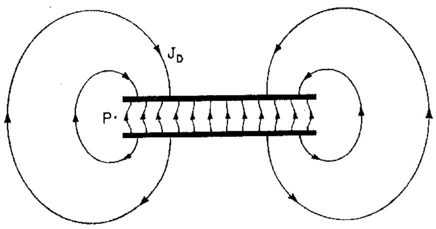

FIG. 4. Effective current distribution due to $\dot{\mathbf{D}}$ in discharge of leaky capacitor.

a real current density, say $\mathbf{J}_D$. Let us then replace $\dot{\mathbf{D}}$ everywhere with $\mathbf{J}_D$ (Fig. 4). We could, in principle, calculate the contribution of the current field $\mathbf{J}_D$ to the magnetic field at P using the Biot–Savart law, Eq. (1). The result is, however, zero; this means that the Biot–Savart contribution from that portion of $\mathbf{J}_D$ which lies "between" the plates is *exactly canceled* by the contribution from the fringing $\mathbf{J}_D$. One way of seeing that the $\mathbf{J}_D$ distribution produces no magnetic field is to recognize that $\mathbf{B}$ is given by[5]

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int\frac{\mathrm{curl}\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau'.$$

Hence if $\mathbf{J}$ in this equation is $\mathbf{J}_D$, and thus expressible, for quasi-steady situations, as the gradient of a scalar, the curl of $\mathbf{J}$ is zero and the integral vanishes. This result is implicit in Planck's statement quoted earlier. Thus all that remains is the Biot–Savart contribution from the original real currents—that is, the motion of the ions.

[5] See, for example, S. J. Raff, Am. J. Phys. **26**, 454 (1958).

5. CONCLUDING REMARKS

It is not at all our intention to seek to diminish the importance of the displacement current in electromagnetic theory. In particular, the treatment of electromagnetic waves would be absurdly complicated if the fields were always referred back to the motions of real charges. And even in many circuit problems it is much simpler to compute magnetic fields from the circuital theorem than from the Biot–Savart law.

We have, however, sought to make two main points. The first is to emphasize that, in all cases of quasi-steady currents, be they closed or unclosed, the magnetic field can be calculated by applying the Biot–Savart law to all the moving charges in the system. The second is that, in calculating by the alternative method of the circuital theorem, in which both conduction currents and displacement currents are included, one must be sure to avoid approximations that are basically incompatible with the special features of the system being investigated.

ACKNOWLEDGMENTS

It is a pleasure to thank Professors Nathaniel H. Frank and John G. King for helpfully provocative discussions.