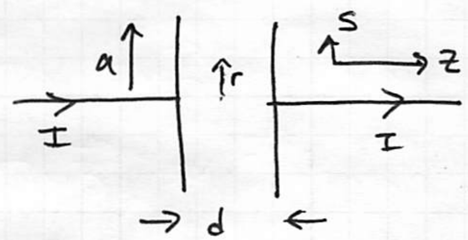
Example: Charging Up a Capacitor

Consider a large parallel plate capacitor made of two metal circular plates (radius, a) separated by a distance d ($d \ll a$). Current runs through the circuit charging the plates. On the positive plate the charge increases $Q(t) = Q_0 + Bt$.
[the linear relationship here is just a model not always true as it depends on $I(t)$]



We aim to determine the magnetic field produced by the changing electric field between the plates.
(we will neglect fringe effects)

The electric field in the plates increases as Q does,

$$\vec{E} = \frac{Q(t)}{A\varepsilon_0}\hat{z}$$

What is the direction of $\vec{B}$?

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{d\vec{E}}{dt}$$

$\vec{J} = 0$,

$$\nabla \times \vec{B} = \varepsilon_0 \mu_0 \frac{d\vec{E}}{dt}$$

$$\iint_S (\nabla \times \vec{B}) \cdot d\vec{a} = \oint_C \vec{B} \cdot d\vec{l} = \varepsilon_0 \mu_0 \iint_S \frac{d\vec{E}}{dt} \cdot d\vec{a}$$

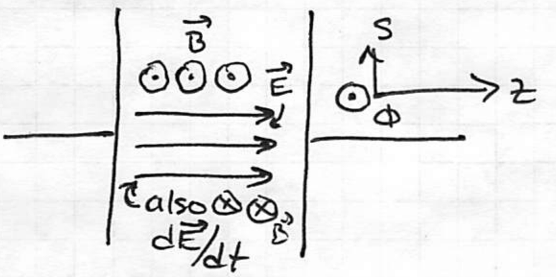
If we go back to thinking about Ampere's Law, we can see $\vec{B}$ circulates around $d\vec{E}/dt$ (like it does w/ $\vec{J}$.)

$$\oint_C \vec{B}\cdot d\vec{l} = \iint_S \mu_0 \varepsilon_0 \frac{d\vec{E}}{dt}\cdot d\vec{A}$$

From this, we ~~...~~ expect that magnetic field to curl around $d\vec{E}/dt$.

As the direction of $\vec{E}$ remains unchanged, only its magnitude increases in this case, $\vec{E}(t) = \frac{Q(t)}{A\varepsilon_0}\hat{z}$, the magnetic field circulates around the electric field. $\pm\hat{\phi}$.

Which direction does it circulate? $+\hat{\phi}$ or $-\hat{\phi}$?



b/c $Q(t)$ increasing.

Use the right hand rule, like we did with $\vec{J}$.

Careful: b/c $d\vec{E}/dt$ matters not $\vec{E}$!

So $\vec{B}$ points in the $+\hat{\phi}$ direction inside the capacitor plates.

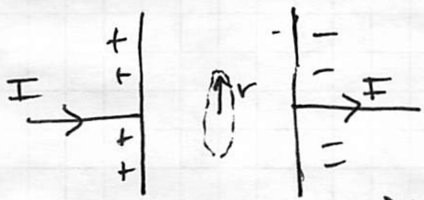
How do we calculate $\vec{B}$?

Because $\vec{E}$ changes the same way everywhere in the capacitor region, we expect $\vec{B}$ to be translationally invariant in z. also, we expect that rotating the system in ϕ has no effect → azimuthal symmetry so ϕ can't matter either. Thus,

$$\vec{B}(s,\phi,z) = B(s)\hat{\phi}$$

This helps us choose a loop to use.

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \varepsilon_0 \iint_S d\vec{E}/dt \cdot d\vec{A}$$



we choose a circular loop that is plane parallel to the plates. $d\vec{A} = s\,d\phi\,ds\,\hat{z}$

with $\vec{E}(t) = \frac{Q(t)}{A\varepsilon_0}\hat{z}$, $\frac{d\vec{E}}{dt} = \frac{dQ/dt}{A\varepsilon_0}\hat{z}$

so that $(Q(t) = Q_0 + \beta t)$,

$$\frac{d\vec{E}}{dt} = \frac{\beta}{A\varepsilon_0}\hat{z}$$

thus,
$$\oint \vec{B}\cdot d\vec{l} = \mu_0\varepsilon_0 \iint_S \frac{\beta}{A\varepsilon_0}\hat{z}\, s\,d\phi\,ds\,\hat{z}$$

$$B 2\pi r = \frac{\mu_0 \varepsilon_0 \beta}{A\varepsilon_0}(\pi r^2)$$

so $\vec{B} = \frac{\mu_0 \beta}{\pi a^2}\frac{r}{2}\hat{\phi}$ check units & limits

$$[B] = [T] = \frac{[\mu_0][\beta][r]}{[a^2]} = \frac{[N/A^2][A][m]}{[m^2]}$$

$$= \frac{N}{mA} = T \checkmark$$

as $r \to 0$, $\vec{B} \to 0$ no enclosed $\vec{E}$ flux ✓