Linearly Magnetized Materials (Recall: Linear Polarization)

Many (common) materials magnetize proportional to the magnetic field.

(Recall: electric polarization $\vec{P} = \varepsilon_0 \chi_E \vec{E}_{tot}$ linear dielectrics)

Like χ_E, we define χ_m in terms of the magnetization but in terms of $\vec{H}$ (not $\vec{B}$!).

$$\vec{M} = \chi_m \vec{H}$$

χ_m is the "magnetic susceptibility" it's unitless and small!

[Note the lack of "symmetry" here, we defined χ_e in terms of $\vec{E}$, but χ_m is defined in terms of $\vec{H}$ not $\vec{B}$. Why? $\vec{H}$ is easy to compute usually. $\oint \vec{H} \cdot d\vec{l} = I_{free}$ easy to measure!]

Unlike χ_E, which was always positive, χ_m can be positive or negative,

$\chi_m > 0$ for paramagnetic materials, $\vec{M}$ lines up w/ $\vec{H}$

$\chi_m < 0$ for diamagnetic materials, $\vec{M}$ opposes $\vec{H}$

With $\vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M} \Rightarrow \vec{B} = \mu_0 (\vec{H} + \vec{M})$

so $\vec{B} = \mu_0 \vec{H} (1 + \chi_m)$

So $\vec{H}$ & $\vec{B}$ point in the same direction if $\chi_m > 0$ and if $\chi_m < 0$ as long as $|\chi_m| < 1$.

Typically, $|\chi_m| \sim 10^{-5} - 10^{-4}$

superconductors? $\chi_m = -1$ ($\vec{B} = 0$ inside, total shielding!)

To summarize!

$$\vec{M} = \chi_m \vec{H}$$

$$\vec{B} = \mu_0 (1+\chi_m) \vec{H} \equiv \mu \vec{H}$$

permeability

$$\mu = \mu_0 (1+\chi_m)$$

$$\vec{M} = \chi_m / \mu \, \vec{B}$$

In free space, $\vec{B} = \mu_0 \vec{H}$ so $\mu_0 = 4\pi \cdot 10^{-7} \frac{Tm}{A}$ is the "permeability of free space."

Let's return to the Aluminum rod where,

$$\chi_{Al} = +2 \cdot 10^{5}$$

paramagnetic $\chi_m > 0$

$$\vec{H}_{inside} = \frac{I}{2\pi R^2} s \hat{\varphi} \quad \text{is what we found}$$

so

$$\vec{M} = \chi_m \vec{H} = \chi_m \frac{I}{2\pi R^2} s \hat{\varphi} \quad (\text{very small}, 10^{-5} \text{ lower})$$

$$\vec{B}_{inside} = \frac{\mu}{\chi_m} \vec{M} = \frac{\mu I}{2\pi R^2} s \hat{\varphi}$$

Because $\mu = \mu_0 (1+\chi_m)$, small

the field inside is enhanced a little bit, which we expect because Al is paramagnetic.

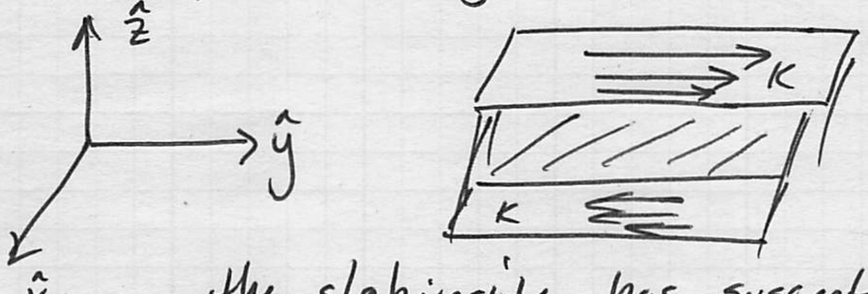
Copper has $\chi_m = -10^{-5}$, so all the equations are the same but now $\chi_m < 0$ so $\vec{B}_{inside}$ is a little suppressed

Outside? None of this matters!

$$\vec{M} = 0!$$

Example: Material between two sheets of current

Consider two sheets (infinite extent in x + y) carrying opposing surface currents, magnitude $|K|$.



A linear material is placed between them that fills the space.

the slab inside has susceptibility, χ_m

on top: $\vec{K} = K\hat{y}$ on bottom: $\vec{K} = -K\hat{y}$

What is $\vec{B}$, $\vec{H}$, & $\vec{M}$ in the slab?

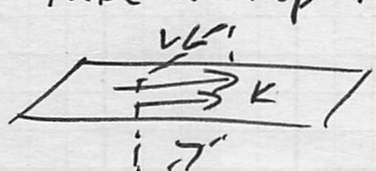
Finding H is usually the simplest because of Ampere's Law,

$$\oint \vec{H}\cdot d\vec{l} = I_{free}$$

this is given by $\vec{K}$ b/c we placed it!

Thinking about Biot-Savart, I can see that $\vec{H}$ points in $-\hat{x}$! between the slabs. (It cancels outside!)

Take a loop that includes one sheet. (Why? Two sheets means $I_{enc} = 0$)



$$\oint \vec{H}\cdot d\vec{l} = HL = I_{enc} = KL$$

So, $\vec{H} = \begin{cases} -K\hat{x} & \text{between sheets} \\ 0 & \text{outside} \end{cases}$

inside the slab, $\vec{M}_m = \chi_m \vec{H} = -\chi_m K\hat{x}$

$$\vec{B}_{inside} = \mu\vec{H} = -\mu K\hat{x} = -\mu_0(1+\chi_m)K\hat{x}$$

very close to free space; just a little enhancement (para) or reduction (dia)

What do the bound currents look like? (Recall these are in and on the material)

$\vec{M} = -\chi_m K \hat{x}$

inside: $\nabla \times \vec{M} = 0$ (uniform $\vec{M}$)

so there's no bound volume currents!

But $\vec{M} \times \hat{n} = \vec{K}_B \neq 0$ $\quad \hat{n} = \begin{cases} +\hat{z} & \text{on top} \\ -\hat{z} & \text{on bottom} \end{cases}$

$$\vec{K}_B = \vec{M} \times \hat{n} = \begin{cases} -\chi_m K(\hat{x} \times +\hat{z}) = +\chi_m K \hat{y} \\ -\chi_m K(\hat{x} \times -\hat{z}) = -\chi_m K \hat{y} \end{cases}$$

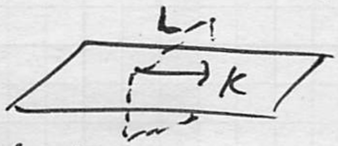
Both Bound surface currents are parallel to K_{free}, but small, if $\chi_m > 0$. $B\uparrow$

if $\chi_m < 0$, the oppose K_{free} $\quad B\downarrow$

Boundary Conditions on $\vec{H}$

With $\oint \vec{H} \cdot d\vec{l} = I_{free}$, $H_{\parallel}$ is discontinuous at boundaries

it is discontinuous by an amount proportional to the surface current (the free surface current)



$$H^{\parallel}_{above} L - H^{\parallel}_{below} L = K_{free} L$$

Notice: $H^{\parallel}$ has two possible components parallel to K and $\perp$ to it but in the plane of K. So strictly speaking

$$\vec{H}^{\parallel}_{above} - \vec{H}^{\parallel}_{below} = \vec{K}_f \times \hat{n}$$

(convince yourself of the direction here!)

What about $H_\perp$?

$$\vec{H} = \frac{1}{\mu}\vec{B} - \vec{M} \quad \text{so that} \quad \nabla\cdot\vec{H} = -\nabla\cdot\vec{M}$$

that implies $$H_\perp^{above} - H_\perp^{Below} = -\left(M_\perp^{above} - M_\perp^{below}\right)$$

The right hand side vanishes if $\vec{M}$ is continuous because $\vec{M} = \frac{\chi_m}{\mu}\vec{B}$ for linear materials and $B_\perp$ is always continuous by $\nabla\cdot\vec{B} = 0$.

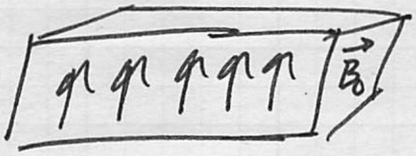
So $H_\perp$ is always continuous everywhere except where χ_m suddenly changes (edge of material)

Consequence: $\oint \vec{H}\cdot d\vec{l} = I_{free}$ looks really simple and for cases of "high symmetry" we can find $\vec{H}$ easily (Ampere's Law). But if symmetry is not high, Be Careful!

for example, if $I_{free} = 0$ everywhere, you cannot (in general) conclude $H = 0$ everywhere! (Toy magnet example) Just b/c $\nabla\times\vec{H} = 0$ everywhere doesn't mean $\vec{H} = 0$ everywhere (unless you can invoke some strong symmetry argument!)

<u>Example</u>: <u>Method to shield external magnetic fields</u>

Consider a really large chunk of material with suseptibility, χ_m. In that material is a uniform magnetic field $\vec{B}_0$ that points upward.



Here, $\vec{B}_0$ is the total magnetic field arising from both the external magnetic field <u>and</u> the magnetization of the material (superposed).

So the material has a uniform magnetization & $\vec{H}$,

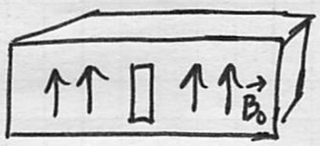
$$\vec{M}_0 = \frac{\chi_m}{\mu}\vec{B}_0 \qquad \vec{H}_0 = \frac{1}{\mu}\vec{B}_0$$

Both uniform upward.

(no matter what χ_m is, $\mu > 0$)

$$\left(\text{also } \vec{H}_0 = \vec{B}_0/\mu_0 - \vec{M}_0\right)$$

In this material, I carve out a small cylindrical hole.



What's $\vec{B}$ & $\vec{H}$ inside the hole? at the center

[there's no material there, so $\vec{M} = 0$ there]

outside the very small hole, H remains relatively unchanged → there were no free currents to begin with so,

There are no free currents at the boundary so $H^{\parallel}_{above} - H^{\parallel}_{below} = 0$

So inside the hole, $\vec{H} = \vec{H}_0 = H_0\hat{z}$

(<u>just like the bulk</u>)

But the magnetic field is, $\vec{B} = \mu_0\vec{H} = \mu_0 H_0 \hat{z}$ inside at center

In terms of B_0 & M_0, $\mu_0 H_0 = B_0 - \mu_0 M_0 = \dfrac{B_0}{1+\chi_m} = B$ in hole

if $\chi_m < 0$ then B_{in} is enhanced
if $\chi_m > 0$ then B_{in} is reduced

} Due to bound currents on the walls - like solenoid.

this kind of setup can shield cavities from magnetic fields (for materials with high χ_m)

[* Remember no fields are blocked, it is the superposition of all the fields that determine the net field anywhere]

Mu-Metal

77% Ni
16% Fe
5% Cu
2% Cr

Mu metal is an alloy that is used to shield against static or low frequency magnetic fields in experimental situations (Passive shield)

Mu metal has very high susceptibility.

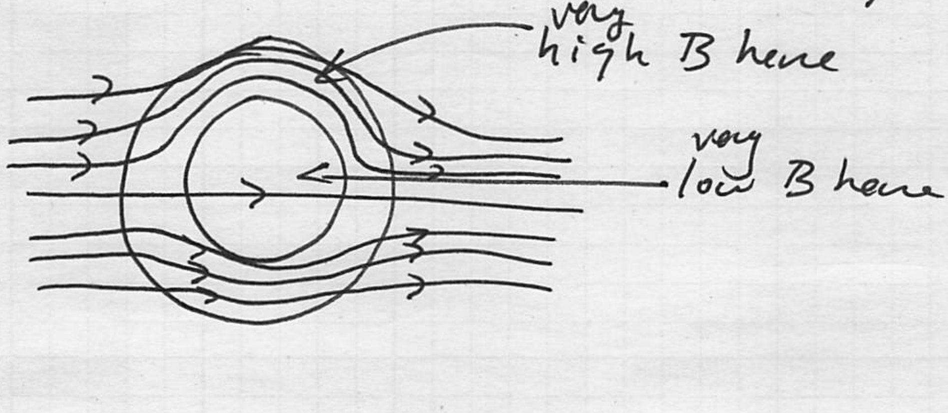
$\mu/\mu_0 = 10^5$ (that's plus 5!)

$\chi_m = +10^5$ it's a super paramagnet

inside mu-metal very high B

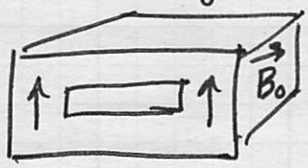
inside the hole, B is very small

Recall $B_{in} = B_0/(1+\chi_m)$ lower by 10^5!

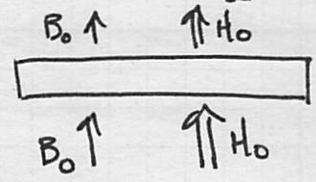


<u>What about a wafer-shaped cavity</u>?

Change the aspect ratio from ▯ to ▭



now



this situation suggests the appropriate boundary condition to make sense of B in the cavity (at center) is,

$$B_\perp^{above} - B_\perp^{below} = 0 \quad \text{so that,}$$

$$\vec{B}_{\substack{at\\center}} = B_0 \hat{z} \quad \text{that is, } \vec{B} \text{ remains unchanged}$$

(instead of $\vec{H}$)

With $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ and $\vec{M} = 0$ in the cavity

we find that, $\vec{H} = \frac{B_0}{\mu_0}\hat{z}$ (which is not $H_0 = \frac{B_0}{\mu}$ the value of H in the material $H = \frac{\mu}{\mu_0} H_0$)

so

$$\vec{H}_{center} = \frac{B_0}{\mu_0}\hat{z} = \frac{\mu}{\mu_0} H_0 \hat{z} = \vec{H}_0 + \vec{M}_0$$

for paramagnetic material, H_{center} is enhanced ($\mu > 1$)

for diamagnetic material, H_{center} is reduced ($\mu < 1$)

[$H_\perp$ <u>can</u> jump at a boundary if $M_\perp$ changes suddenly.]