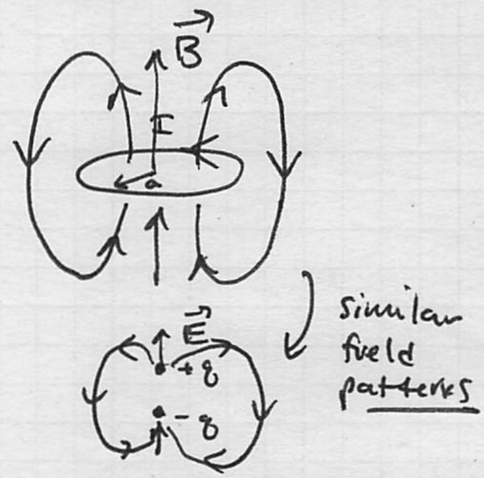
We have developed tools to find $\vec{B}$ (or $\vec{H}$) in general, but like $\vec{E}$ there are very important, specific distributions that are worth more detailed study. For $\vec{E}$, it was the pt. charge and the dipole. For $\vec{B}$ (and $\vec{A}$), it will be the magnetic dipole, that arises from a loop of current.

We will develop a multipole expansion of $\vec{A}$ that will help us investigate this field because Biot-Savart (in general) is a pain.



Magnetic Dipole

Consider a small loop of current. the magnetic field → looks a bit like an electric dipole

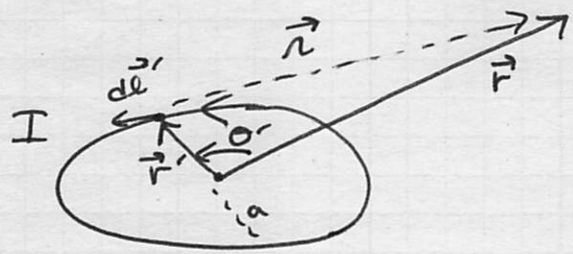
Calculating $\vec{B}$ in general from this distribution using Biot-Savart is a real pain, but this is an important field to know. You might expect this from a classical electron orbit. $\vec{A}$ will help us out and the $\vec{B} = \nabla \times \vec{A}$ will get us $\vec{B}$ and it will be fairly straight forward.

Note: For a finite loop, finding $\vec{A}$ is tough, so we will consider the limit as $a \to 0$.

This is like finding the electric potential far away from the sources. You write a multipole expansion and look at the leading term (or two) to uncover the dominant behavior of $\vec{E}$.

We will do something similar for $\vec{A}$.

Multipole Expansion (for a loop of current)



The setup for the loop of current is to the left.

Recall we will be solving this problem (approximately) as $a \to 0$ or $r \to \infty$ (very far from the loop).

In general,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{\mathscr{r}} \, d\tau'$$

for line currents,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I}}{\mathscr{r}} \, dl' = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}'}{\mathscr{r}} \quad \left(\text{where } d\vec{l}' \text{ points along } \vec{I}. \right)$$

for this problem, the integral is around the loop.

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l}'}{\mathscr{r}}$$

The separation vector's magnitude can be written as,

$$\mathscr{r} = \sqrt{r^2 + r'^2 - 2rr'\cos\theta'}$$

Be careful here θ' is the angle given by $\hat{r} \cdot \hat{r}' = \cos\theta'$. It is not the usual polar angle!

We are solving the problem far away, $r \gg r'$, so expand $\mathscr{r}$,

$$\mathscr{r} \simeq r\left(1 - \frac{r'}{r}\cos\theta' + \mathcal{O}(r'^2/r^2)\right) \quad \text{(we have done this before with V)}$$

so

$$\frac{1}{\mathscr{r}} \simeq \frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta' + \mathcal{O}(r'^2/r^2)\right)$$

so our approximate vector potential far from the loop is,

$$\vec{A}(\vec{r}) \simeq \frac{\mu_0 I}{4\pi r}\left[\oint d\vec{l}' + \oint \frac{r'\cos\theta' \, d\vec{l}'}{r} + \dots \right]$$

this is very similar to the multipole expansion for V, but it is a vector expansion ($d\vec{l}'$)

- each term has an additional $1/r$ term and $P_n(\cos\theta')$ dependence.

$$\vec{A}(\vec{r}) \simeq \frac{\mu_0 I}{4\pi r}\left[\oint d\vec{l}\,' + \oint \frac{r'\cos\theta'}{r}\, d\vec{l}\,'\right]$$

Here we only keep the first two terms: Monopole + dipole.

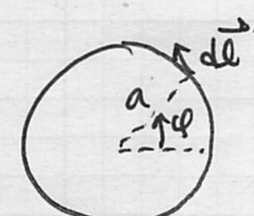
We expect the monopole term to vanish as they are no magnetic monopoles!

Monopole Term

$$\vec{A}_{mono} = \frac{\mu_0 I}{4\pi r}\underbrace{\oint d\vec{l}\,'}_{\text{has to vanish}}$$

is $\oint d\vec{l}\,' = 2\pi R$? No!

From above



Notice

$d\vec{l}\,' = a\,d\varphi' \langle -\sin\varphi, \cos\varphi, 0\rangle$

so $$\oint d\vec{l}\,' = \int_0^{2\pi} a\,d\varphi' \langle -\sin\varphi, \cos\varphi, 0\rangle$$

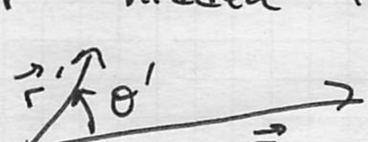
$$= a\left[\langle \cos\varphi, \sin\varphi, 0\rangle\right]_0^{2\pi} = a\left[\langle 0,0,0\rangle\right] = 0 \checkmark$$

Dipole Term

So the dominant term for $\vec{A}$ will almost always be the dipole term (it may vanish, but not for simple wire loops).

$$\vec{A}_{dipole} = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\theta'\, d\vec{l}\,'$$

We can compute this integral, but note that there is an $\vec{r}$ "hidden" in $\cos\theta'$



$r'\cos\theta' = \vec{r}\,' \cdot \hat{r}$

this hidden $\vec{r}$ makes this integral a royal pain in the neck. But if we rewrite the integral,

$$\oint r' \cos\theta' \, d\vec{l}' = \oint \vec{r}' \cdot \hat{r} \, d\vec{l}'$$

we can invoke Stokes' theorem and several vector identities to show,

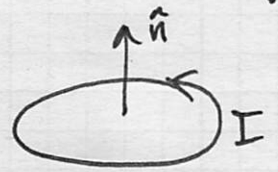
$$\oint \vec{r}' \cdot \hat{r} \, d\vec{l}' = -\hat{r} \times \iint d\vec{a}' = \iint d\vec{a}' \times \hat{r}$$

* This is <u>not</u> an obvious or easy result to derive; it's a great exercise in vector identities and vector calculus.

We will just use the result so let's do that.

<u>Define</u> $\vec{m}$ = magnetic moment of the current loop $\equiv I \iint d\vec{a}'$

for a flat loop,



Area = a

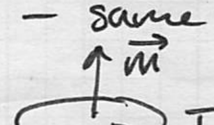
($\hat{n}$ determined by Right hand rule)

just the usual area vector. $\Rightarrow \iint d\vec{a}' = a\hat{n}$ going around loop in same sense of current

With our vector identity, we find

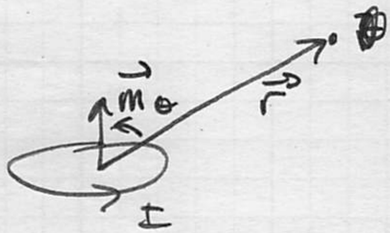
$$\vec{A}_{dipole} = \frac{\mu_0 I}{4\pi r^2} \iint d\vec{a}' \times \hat{r} = \frac{\mu_0}{4\pi r^2} \vec{m} \times \hat{r}$$

The direction of the magnetic dipole moment, $\vec{m}$, is determined by the current direction in the loop & the right hand rule — same direction as area vector.



magnitude, current x area.

What does $\vec{A}$ look like off axis?



Say you are at some location $\vec{r}$, which makes an angle θ (polar angle) with $\vec{m}$. (let $\vec{m} = m\hat{z}$, e.g.)

$$\vec{A} \cong \frac{\mu_0 m}{4\pi r^2} \sin\theta \underbrace{(\hat{z}\times\hat{r})}_{\hat{\varphi}!}$$

so $\vec{A}$ points in the $\hat{\varphi}$ direction like the current! A "follows" the current.

the magnetic potential falls off like $1/r^2$ as we expect for a dipole.

With $\vec{A} \cong \frac{\mu_0 m}{4\pi r^2} \sin\theta\,\hat{\varphi}$ we find that $\vec{B}$ is a dipole field (as expected) [approx.]

$$\vec{B} = \nabla\times\vec{A}$$

$$\vec{B} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\varphi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(r A_\varphi)\hat{\theta}$$

$$\cong \frac{\mu_0 m}{4\pi}\left[\frac{1}{r^3\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)\hat{r} - \frac{\sin\theta}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\theta}\right]$$

$$= \frac{\mu_0 m}{4\pi}\left[\frac{1}{r^3\sin\theta} 2\sin\theta\cos\theta\,\hat{r} - \frac{\sin\theta}{r}\left(\frac{-1}{r^2}\right)\hat{\theta}\right]$$

$$\vec{B} \cong \frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

Dipole field with $1/r^3$ dropoff.