With Biot-Savart we can setup an integral to find the magnetic field due to a distribution of currents. In it's most general form it looks thusly,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

Just like with Coulomb's law, we might find the integral is complicated, difficult, or impossible. And for some problems, it complicated in a way that can simplified by symmetry → coupled with another method of attack – Ampere's Law.

There are two of Maxwell's Equations that tell you the connection between current and magnetic field. This is analogous to $\vec{E}$ and ρ.

$$\nabla\cdot\vec{E} = \rho/\epsilon_0 \longleftrightarrow \nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} = 0 \longleftrightarrow \nabla\times\vec{B} = \mu_0\vec{J} \quad \text{(Ampere's Law)}$$

All these equations describe electro/magneto statics. Time dependence will be introduced in 482.

We could argue that $\nabla\cdot\vec{B}=0$ & $\nabla\times\vec{B}=\mu_0\vec{J}$ are experimental facts, but they are in fact consistent with Biot Savart.

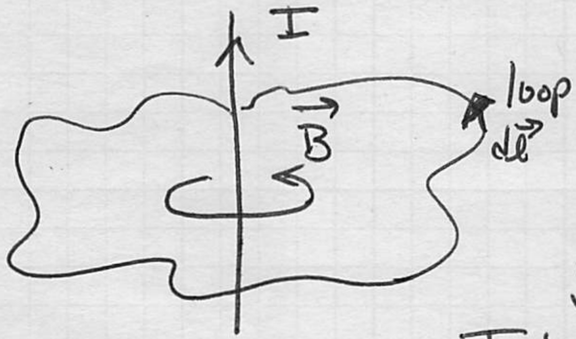
From Biot-Savart → you can show Ampere's Law (HARD)

From Ampere's Law → you can show Biot-Savart (HARDER)

Much like from Coulomb, you can show Gauss and vice versa.

- These equations are very deep and broad in their utility and applicability.

We won't prove the connection between Ampere's Law and Biot Savart, insted we will argue that such a connection is plausible. We will use the case of the infinite wire, which we solved with Biot-Savart,



$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

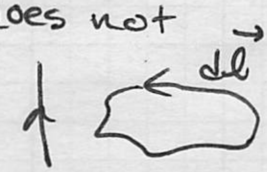
In this case,

$$\underbrace{\oint \vec{B}\cdot d\vec{l}}_{\text{Integral around any arbitrary loop.}} = \frac{\mu_0}{2\pi} I \int \frac{1}{s}\hat{\phi}\cdot d\vec{l}$$

$d\vec{l} = s\,d\phi\,\hat{\phi}$ as we are going around the wire in a plane $\perp$ to the wire.

$$\oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi}\int \frac{1}{s}\hat{\phi}\cdot(s\,d\phi\,\hat{\phi}) = \frac{\mu_0}{2\pi} I \underbrace{\int d\phi}_{2\pi} = \mu_0 I$$

A more careful calcutation that does not enclose I shows $\oint \vec{B}\cdot d\vec{l} = 0 \longrightarrow$



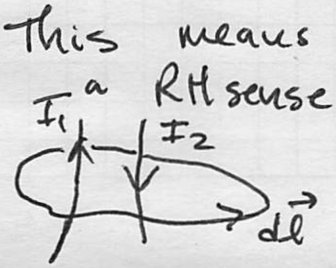
so, $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}$

Through superposition, with multiple currents,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{total poking through loop.}}$$

Note: If loop goes other direction then $d\vec{l}$ points the other way and you'll pick up a minus sign.

This means I is $(+)$ if it "pokes" through in a RH sense and $(-)$ if it does in the LH sense.



$\oint \vec{B}\cdot d\vec{l} = \mu_0(|I_1| - |I_2|)$ here.

As I is defined in terms of current density,

$$I \equiv \int \vec{J} \cdot d\vec{A}$$

we can find how $\oint \vec{B} \cdot d\vec{l}$ is related to $\vec{J}$,

so, $$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{A}$$

But By Stoke's,

$$\oint \vec{B} \cdot d\vec{l} = \int (\nabla \times \vec{B}) \cdot d\vec{A} = \mu_0 \int \vec{J} \cdot d\vec{A}$$

which is true for any loop/area such that,

$$\int (\nabla \times \vec{B} - \mu_0 \vec{J}) \cdot d\vec{A} = 0$$ and thus,

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

this is not a proof as we assumed infinite wires.

the real proof starts with Biot-Savart,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

and taking $\nabla \cdot \vec{B}$ and $\nabla \times \vec{B}$ of this to show $\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{B} = \mu_0 \vec{J}$

It's a lovely exercise in vector calc!
(But not worth our energy/time ☺).

Here's a puzzle! $I_{through} = \int \vec{J} \cdot d\vec{A}$ But there are many surfaces sharing the same boundary loop. Which do you use?

Answer: It's $I_{through}$ so any/all are ok!

Ok, let's apply Ampere's Law to a few steady current situations.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \quad \text{or} \quad \nabla \times \vec{B} = \mu_0 \vec{J}$$

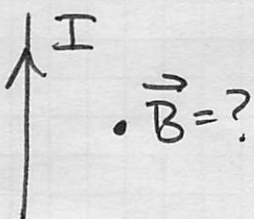
any loop

I "poking" through that loop in the right hand sense.

Like Gauss' Law, Ampere's Law let's us find $\vec{B}$ if we have sufficient symmetry and we can argue up front that B is constant and/or pulled out of the integral (like Gauss).

★ Ampere's Law (in magnetostatics) is <u>always</u> <u>true</u>, but <u>not</u> <u>always</u> <u>useful</u> (like Gauss!).

<u>Example 1</u>: <u>Infinite Wire</u>



We need to first argue we can use Ampere's Law here.

1) $\vec{B}$ cannot be radial → violates $\nabla \cdot \vec{B} = 0$ or $\int \vec{B} \cdot d\vec{A} = 0$
2) $\vec{B}$ cannot depend on z or φ (symmetry!)
3) $\vec{B}$ cannot have a $\hat{z}$ component! (Biot-Savart, $d\vec{l} \times \hat{r}$)

so $\quad \vec{B}(\vec{r}) = B(s)\hat{\varphi}$ so that,

$$\oint \vec{B} \cdot d\vec{l} = B\int dl = B 2\pi s = \mu_0 I \quad \text{so}$$

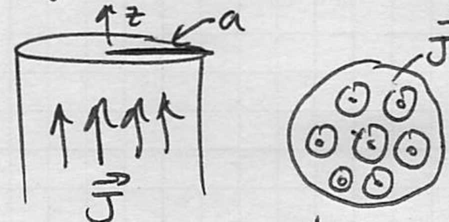
$$B = \frac{\mu_0 I}{2\pi s} \quad \text{or} \quad \vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi s}\hat{\varphi}$$

So much easier than Biot-Savart!

Example 2: Thick Wire

Consider a wire of radius, a, with a uniform current density, $\vec{J}$ inside the wire.

$$\vec{J} = J_0 \hat{z} \quad \text{for } s \leq a$$
$$= 0 \quad \text{for } s > a$$



side

top view

We draw an Amperian loop (circle) centered at the origin,

Like the thin wire (example 1),

$\vec{B}(\vec{r}) = B(s)\hat{\phi}$ → nothing changed to our 3 arguments by thickening the wire!

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$$

$s \leq a$,
$$\oint \vec{B} \cdot d\vec{l} = B \int dl = B 2\pi s \quad \text{as before}$$

$$\mu_0 I_{enclosed} = \mu_0 \int \vec{J} \cdot d\vec{A} = \underbrace{\mu_0 J_0 \pi s^2}$$

Because we only enclosed current up to s.

So,
$$B 2\pi s = \mu_0 J_0 \pi s^2 \Rightarrow \vec{B} = \mu_0 J_0 \frac{s}{2} \hat{\phi}$$

for $s > a$,

$$\oint \vec{B} \cdot d\vec{l} = B 2\pi s \quad \text{as before}$$

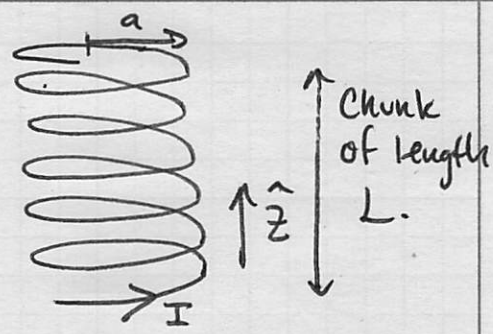
But now, we have enclosed all the current up to $s = a$.

$$\mu_0 I_{enclosed} = \mu_0 \int \vec{J} \cdot d\vec{A} = \mu_0 J_0 \pi a^2$$

So,
$$\vec{B} = \frac{\mu_0 J_0 a^2}{2s} \hat{\phi}$$

Note: no surface current
B is continuous
and $B(0) = 0$

Example 3: Infinite Solenoid



Consider a solenoid of radius, a, with n turns per meter.
(in the sketch $nL = 6$)

Let's see if we can argue to use Ampere's Law in this case.

- First off, $\vec{B}$ better not have a component is the $\hat{s}$ direction as that violates $\nabla \cdot \vec{B} = 0$.
- Could $\vec{B}$ have a $\hat{\phi}$ component? It's tempting to think so because the long wire gave that. But draw a loop around the solenoid $\perp$ to z direction and no current pokes through, it circles around, which suggest that $\vec{B}$ should have no $\hat{\phi}$ component (Biot-Savart)

(an aside: we assume we really have $\vec{K}$ purely in $\hat{\phi}$ direction. In reality I does flow up the page and you get a small contribution due to this. But it's very small compared to the effect of "$\vec{K}$" in $\hat{\phi}$.)

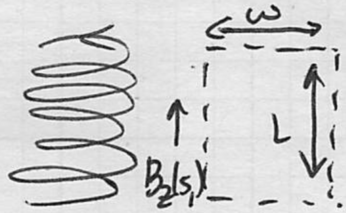
So it appears $\vec{B}(\vec{r}) = B(s)\hat{z}$ is what we have.

Note:
$B_z(s \to \infty) = 0$
- far from currents we expect no B field.
- Also Biot-Savart $1/r^2$
- Also observations.

How do we construct an Amperian loop to handle calculating $\vec{B}$ in this setup?

If $\vec{B}(\vec{r}) = B(s)\hat{z}$, then the loop must capture both the $\hat{z}$ nature of the field and s dependence. So we use a loop $\perp$ to the direction of the current in the solenoid that includes both a bit of length in z and s.

Outside the solenoid,



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \underline{\underline{I_{enc}}} = 0$$

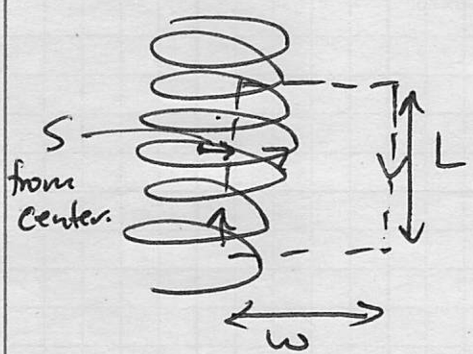
no enclosed current!

$$\oint \vec{B}\cdot d\vec{l} = (B(s_1) - B(s_2))L = 0$$

so, $B(s_1) = B(s_2)$ everywhere outside.

with $B(s\to\infty) = 0$ then $B = 0$ everywhere outside.

Inside the solenoid,



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}$$

$$B(s)L + 0 = \mu_0 I n L$$

so $\vec{B} = \mu_0 I n \hat{z}$ inside

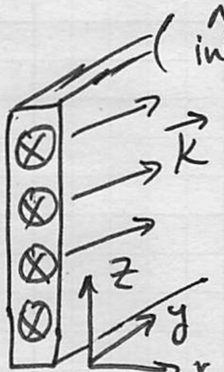
uniform magnetic field in $\hat{z}$

$$\vec{B} = \begin{cases} \mu_0 n I \hat{z} & s < a \\ 0 & s > a \end{cases}$$

(Kind of like a capacitor analog for $\vec{B}$)

Example 4: Current Sheet

Consider a sheet of current flowing thusly

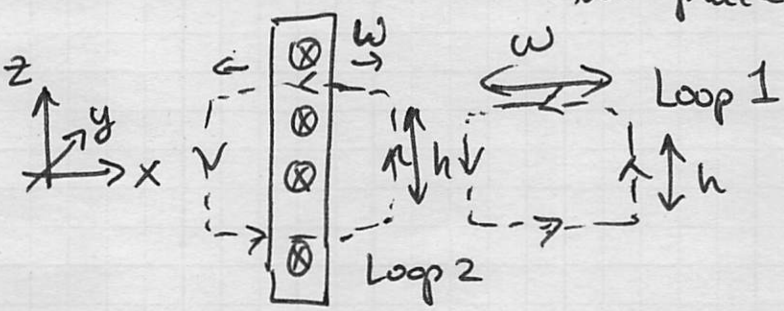


Let's find the magnetic field on each side of the sheet.

(this example will help us understand the boundary conditions associated with $\vec{B}$.)

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

we will use Ampere's Law in two places.



We expect $\vec{B}(\vec{r}) = B(x)\hat{z}$.

We start with Loop 1 where $I_{enc}=0$

$$\oint \vec{B}\cdot d\vec{l} = B(x_2)h - B(x_1)h = 0$$

$$B(x_2) = B(x_1) \quad \text{can't } B(\vec{r}\to\infty) = 0?$$

no b/c current $\neq 0$ @ $\vec{r}\to\infty$

so, what about loop 2?

$$\mu_0 I_{enc} = \mu_0 K h$$

$$\oint \vec{B}\cdot d\vec{l} = B_{right}h + B_{left}h = 2Bh = \mu_0 K h$$

why? B changes direction!

why? $|\vec{B}|$ doesn't depend on location!

so, $B = \frac{\mu_0 K}{2}$ rather, $\vec{B}_{right} = \frac{\mu_0 K}{2}\hat{z}$

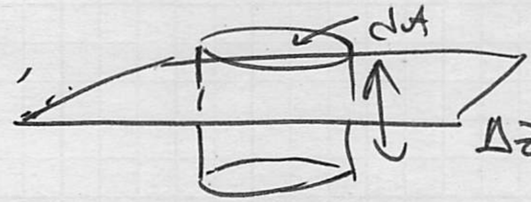
$$\vec{B}_{left} = -\frac{\mu_0 K}{2}\hat{z}$$

There's something very useful about this example: it clearly illustrates the Boundary Conditions on $\vec{B}$.

Boundary Conditions for $\vec{B}$

Consider Gauss' Law for $\vec{B}$, $\nabla\cdot\vec{B} = 0$

If we confine the current sheet by a pill box,



and let $\Delta z \to 0$

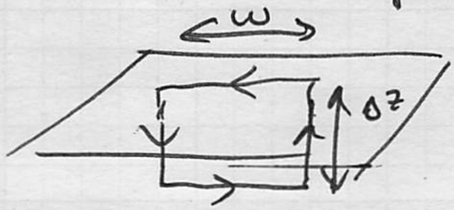
then $\oint_A \vec{B}\cdot d\vec{A} \to \vec{B}_{top}\cdot d\vec{A} + \vec{B}_{bottom}\cdot d\vec{A} = 0$

thus,

$B_{top\perp} = B_{bottom,\perp}$ that is the magnetic field $\perp$ to the boundary is <u>continuous</u>.

Consider Ampere's law for $\vec{B}$, $\nabla\times\vec{B} = \mu_0\vec{J}$

if we use an Amperian Loop on the surface,



and let $\Delta z \to 0$

then $\oint \vec{B}\cdot d\vec{l} = \vec{B}_{above}\cdot\vec{w} + \vec{B}_{below}\cdot\vec{w} = \mu_0 K w$

so that,

$$B_{above,\parallel} - B_{below,\parallel} = \mu_0 K$$

So the parallel component of the magnetic field is discontinuous by an amount ~~given~~ proportional to the surface current.

Griffiths summarizes this discontinuity thusly,

$$\vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{K} \times \hat{n})$$

where $\hat{n}$ is the vector normal to the surface.