We uncovered the continuity equation, which tells us how charges move and build up,

$$\frac{d\rho}{dt} + \nabla \cdot \vec{J} = 0$$

For this class, we are concerned with timeindependent situations. that is where charges don't pile up anywhere, i.e., $\frac{d\rho}{dt} = 0$

So for us, $\nabla \cdot \vec{J} = 0$ defines ~~magnetostatic~~ magnetostatic situations.

$\nabla \cdot \vec{J}$ suggest that current does not diverge, and thus we expect steady currents. No charge builds up and there is no time dependence in any magnetostatic situation.

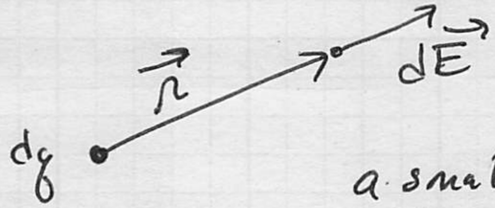
Currents create Magnetic Fields

- This is an experimental fact. We have experiments that determine direction and magnitude. I can't derive this.

- Later, we will see the formula as two of Maxwell's Equations, but Biot & Savart deduced their famous equation from careful experimentation following Oersted's original discovery.

Let's use an analogy,

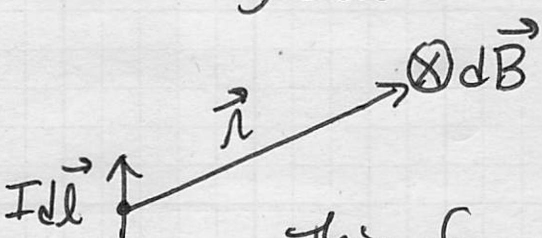
From Coulomb's law we know we can find the contribution, $d\vec{E}$, to the electric field due to a small chunk of charge, dq.



$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\hat{r}$$

a small charge, dq, produces a small field, $d\vec{E}$.

Biot-Savart is quite similar. We can find the small contribution to the magnetic field, $d\vec{B}$, due to a small segment of current, $\vec{I}dl$.



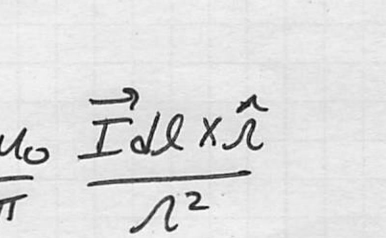
$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{\vec{I}dl \times \hat{r}}{r^2}$$

This formula is quite similar to Coulomb except for the cross product.

Of course, there are no isolated chunks of current, if they were we wouldn't have steady currents.

So we have to sum over the chunks.

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}dl'}{r^2} \times \hat{r}$$



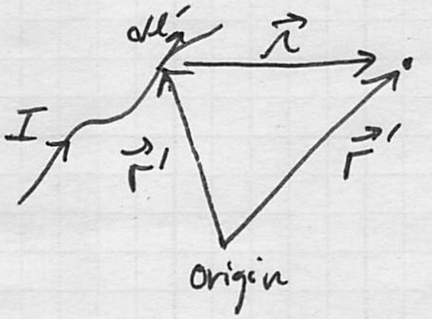
the permeability of free space (magnetic partner to ϵ_0)

$$\frac{\mu_0}{4\pi} = 10^{-7}\ N/A^2 \text{ is SI units.}$$

Some Authors prefer that I not be a vector and shift the vector to $d\vec{l}$, which points in the same direction as I. So you might see,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int I \frac{d\vec{l}' \times \hat{\imath}}{\imath^2}$$ in some places.

Remember that $\vec{\imath}$ is the relative position vector,

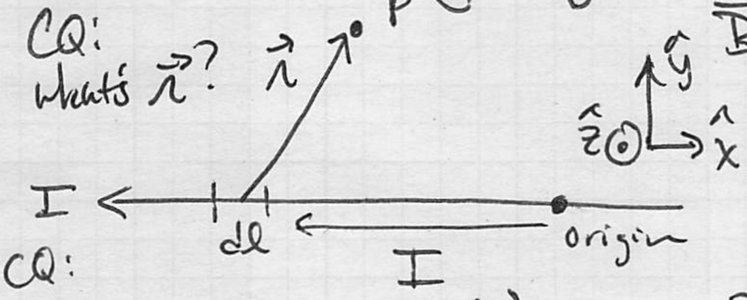


$$\vec{\imath} = \vec{r} - \vec{r}'$$

Example: Line of current

We will try to construct $d\vec{B}$ for a line of current that runs in the $-x$ direction and then find the field when the line is infinitely long.

CQ: what's $\vec{\imath}$?



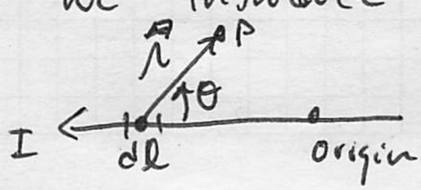
$\vec{B}(\vec{r}) = ?$

Let's start with $d\vec{B}$ due to dl,

CQ:
Which way does $d\vec{B}$ point?

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}' \times \hat{\imath}}{\imath^2} = \frac{\mu_0}{4\pi}\frac{I\,|d\vec{l}' \times \hat{\imath}|\,(-\hat{z})}{\imath^2}$$

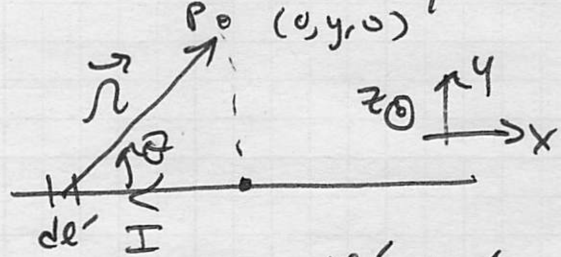
If we introduce an angle θ like so,



how big is $\frac{d\vec{l} \times \hat{\imath}}{\imath^2}$?
CQ.

$$\left|d\vec{l}' \times \hat{\mathscr{r}}\right| = dl' \sin\theta \quad \text{so} \quad \left|\frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}\right| = \frac{dl' \sin\theta}{\mathscr{r}^2}$$

Ok let's construct $d\vec{B}$ in the coordinate system that we have,



$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

$dl' = dx'$

CQ!

$$\frac{I\, d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} \; ?$$

$$d\vec{B} = \frac{\mu_0}{4\pi}(-\hat{z}) \frac{I\, y\, dx'}{(x'^2+y^2)^{3/2}} \Leftarrow$$

we found everything except $\sin\theta$ which is $\frac{y}{\sqrt{x'^2+y^2}}$

So if this is an infinite line (i.e. $\pm\infty$ for x limits), we can solve this fully,

$$\vec{B} = \int d\vec{B} = \frac{\mu_0}{4\pi} I y (-\hat{z}) \int_{-\infty}^{\infty} \frac{dx'}{(x'^2+y^2)^{3/2}}$$

$$= \frac{\mu_0}{4\pi} I y (-\hat{z}) \left[\frac{x'}{y^2\sqrt{x'^2+y^2}}\right]_{-\infty}^{\infty}$$

Uh oh! we get some indeterminate form! well $x >> y$ in this case when it gets evaluated so,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I y (-\hat{z}) \left[\frac{x'}{y^2 |x'| \sqrt{1 + y^2/x^2}}\right]_{-\infty}^{\infty}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I}{y}(-\hat{z})\left[\frac{x'}{|x'|}\underbrace{\overrightarrow{\sqrt{1+y^2/x'^2}}}\right]_{-\infty}^{\infty}$$

this term will evaluate to 1 regardless of $\pm\infty$.

$$= \frac{\mu_0}{4\pi}\frac{I}{y}(\hat{z})\left[\frac{x'}{|x'|}\right]_{-\infty}^{+\infty}$$

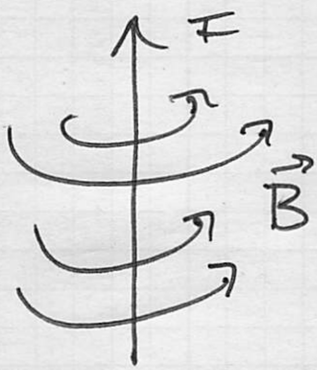
$$= \frac{\mu_0}{4\pi}\frac{I}{y}(-\hat{z})\left[\underbrace{\lim_{x'\to\infty}\frac{x'}{|x'|}} - \underbrace{\lim_{x'\to-\infty}\frac{x'}{|x'|}}\right]$$

this approaches 1   this approaches -1

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I}{y}(-\hat{z})\left[1-(-1)\right]$$

$$\vec{B}(\vec{r}) = -\frac{\mu_0 I}{2\pi y}\hat{z}$$

whew! we will find a short cut later. that makes use of the structure of the field,

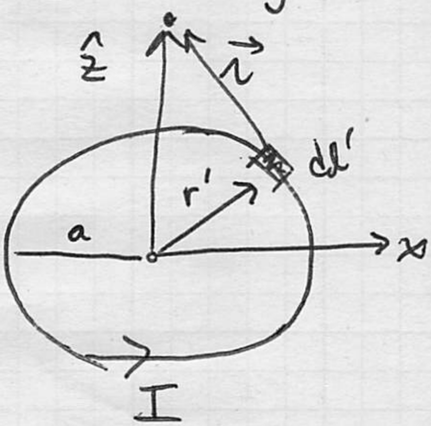


All the contributions to the magnetic field in this example were the same; sometimes they aren't.

CQ: $\vec{B}$ directions vs. $d\vec{B}$.

Example! Ring of Current, determine $\vec{B}$ on the axis?

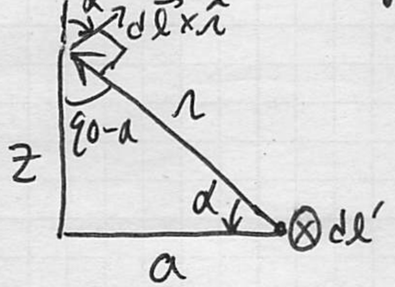
$$\vec{B}(\vec{r}) = \int d\vec{B} = \int \frac{\mu_0}{4\pi} \frac{I\, d\vec{l}' \times \vec{\mathscr{r}}}{\mathscr{r}^2}$$



Notice that $\mathscr{r} = \sqrt{a^2+z^2}$ is a constant, which makes the integral pretty easy!

CQ: $\left|\frac{dl \times \hat{\mathscr{r}}}{\mathscr{r}^2}\right|$?

Unfortunately $d\vec{l} \times \hat{\mathscr{r}}$ points at some crazy angle. But if we sum over all dl's only the vertical component survives.



$|d\vec{l} \times \hat{\mathscr{r}}| = dl'$

But the vertical component is $dl' \cos\alpha = dl' \frac{a}{\sqrt{z^2+a^2}}$

$$dB_z = \frac{\mu_0}{4\pi} \frac{(d\vec{l}' \times \hat{\mathscr{r}})_z}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi} \frac{1}{a^2+z^2} \frac{a\, dl'}{\sqrt{a^2+z^2}}$$

CQ: what is dB_z?

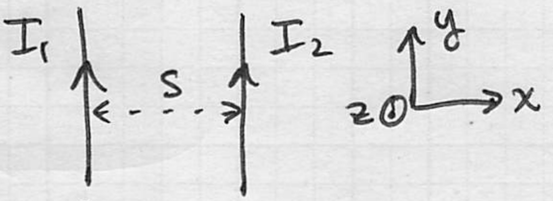
$$B_z = \frac{\mu_0 I}{4\pi} \frac{a}{(a^2+z^2)^{3/2}} \underbrace{\int dl'}_{2\pi a}$$

circumference of the circle.

$$B_z(0,0,z) = \frac{\mu_0 I}{2} \frac{a^2}{(a^2+z^2)^{3/2}}$$

As wires with current create magnetic fields, moving charges (or rather other wires), experience forces near those wires.

For example, consider two parallel wires, separated by a distance, s.



$$\vec{B}_{\substack{\text{due to}\\ I_1}} = \frac{\mu_0 I_1}{2\pi s}(-\hat{z})$$

the force on wire 2 due to the field generated by wire 1 is,

$$\vec{F}_{\text{on 2 by 1}} = \int I_2 \, \underbrace{d\vec{l}_2 \times \vec{B}}_{-x \text{ direction}} = I_2 \frac{\mu_0 I_1}{2\pi s} \int dl_2 (-\hat{x})$$

So the force per unit length is,

$$\frac{F_{\text{on } I_2}}{\text{length}} = \frac{\mu_0 I_1 I_2}{2\pi s} \begin{pmatrix} \text{towards } I_1 \text{ for } \uparrow\uparrow \text{ currents} \\ \text{away } I_1 \text{ for } \uparrow\downarrow \text{ currents} \end{pmatrix}$$

We've looked at currents due to wires and how Biot-Savart predicts the magnetic field that is generated, but this holds for surface and volume currents, too.

$$\vec{B} = \int d\vec{B} = \int \frac{\mu_0}{4\pi} \frac{\vec{K} \times \hat{\imath}}{\imath^2} \, da' \quad \text{for surface currents}$$

$$\vec{B} = \int d\vec{B} = \int \frac{\mu_0}{4\pi} \frac{\vec{J} \times \hat{\imath}}{\imath^2} \, d\tau' \quad \text{for volume currents}$$

Finally superposition works for $\vec{B}$ as it did for $\vec{E}$

$$\vec{B}_{total} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \cdots$$