Energy

- As it turns out, we can use electric fields to do work on charges. This should be fairly obvious as charges in an electric field will experience a force, $\vec{F} = q\vec{E}$.

- Let's see if we can find what kind of work is done on charges and how it's related to things we already know.

Recall that we "invented" $V(\vec{r})$ = electric potential or voltage

- Given $\vec{E}$, we can compute V,

$$V(\vec{r}) = -\int_{\substack{\text{origin where}\\ V=0}}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

- Then we showed (with maths), given ρ, we can compute V,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}}$$

- And further once you have V, you know $\vec{E}$,

$$\vec{E} = -\nabla V$$

and once you have V, you know ρ,

$$\nabla^2 V = -\rho/\epsilon_0$$

<u>But what is</u> <u>V</u> <u>physically</u>? <u>What does it mean?</u>

Consider moving a tiny charge q through electric fields from a to b. In this case,

$\vec{F}_{electric} = q\vec{E}$, so you exert a force

$\vec{F}_{you} = -q\vec{E}$ as you "fight the field"

To move the charge from a to b, you do external work,

$$W_{\substack{ext\\a\to b}} = +\int_a^b \vec{F}_{you}\cdot d\vec{l} = -q\int_a^b \vec{E}\cdot d\vec{l}$$

But the term we integrate is the potential difference between location b and location a!

$$W_{\substack{ext\\a\to b}} = q\Delta V_{ab} = q\left(V(b)-V(a)\right)$$

* So, the voltage carries some information about work & energy!

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Flashback

In 184, if you do work, we can talk about stored potential energy (* caveat: for conservative forces, which Felec is)

So we ~~will~~ had defined the electrostatic potential energy

$PE \equiv qV$ (but we will follow Griffiths')

(Note there's always ambiguity as we can define $PE = 0$ anywhere!)

so, $V(\vec{r}) = PE/q = \dfrac{\text{the potential energy}}{\text{unit charge}}$

[So we could call the potential energy $= U(\vec{r}) = qV(\vec{r})$

But Griffiths calls it W, it's the work needed by you to get q to the point r, which is what we will do.]

+ $qV(\vec{r})$ is the "potential energy of charge q in the prescence of others."

+ But what is the work it takes to get the others together?

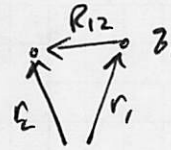
We will derive the "stored electrostatic energy of a system" by building up a configuration of charges one-by-one and calculating the work.

① Bring in q_1. There are no other charges, so no work is done.

$W_1 = 0$

② Bring in q_2. q_1 is already there, producing an electric field.

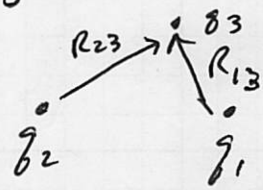
so $W_2 = q_2 V_{\text{caused by } q_1}$



$$= q_2 \left(\frac{1}{4\pi\epsilon_0} \frac{q_1}{R_{12}} \right)$$

③ Now bring in q_3. Both q_1 & q_2 are there producing electric fields

$W_3 = q_3 V_{\text{caused by } q_1 \text{ \& } q_2}$



$$= q_3 \frac{1}{4\pi\epsilon_0} \left(\frac{q_1}{R_{13}} + \frac{q_2}{R_{23}} \right)$$

Total Work done so far: $W_1 + W_2 + W_3$

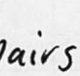
$$W_{system} = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{R_{12}} + \frac{q_1 q_3}{R_{13}} + \frac{q_2 q_3}{R_{23}} \right)$$

There's a pattern developing that can be extended to any number of charges:

Add all the pairs $\frac{q_i q_j}{R_{ij}}$

- But don't compute "self energy" $(i=j)$
- And don't double count!

<u>Clicker Questions</u>



Or....

You <u>can</u> double count and then just divide by 2, this actually helps us derive the result for continuous distributions!

So,
$$W_{system} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_i q_j}{R_{ij}}$$
(Note: this could be negative!)

We can perform a little reorganization,
$$W_{system} = \frac{1}{2}\sum_{i=1}^{n} q_i\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{R_{ij}}\right)$$

that thing in the parentheses looks like a potential! Let's call it $\tilde{V}_i(\vec{r}_i)$; it's the potential you get at point "i" due to <u>all</u> the other charges at <u>all</u> points $j\neq i$ (Be careful not to include "self energies")

So,
$$W_{sys} = \frac{1}{2}\sum_{i=1}^{n} q_i\tilde{V}_i(\vec{r}_i)$$
⇐ We don't need to bring in q's one at a time but we do have to be careful about our counting.

↑ recall: this one-half let's us double count (e.g. 1+2 and 2+1)

$W_{sys} = \frac{1}{2}\sum_{i=1}^{n} q_i\tilde{V}_i(\vec{r}_i)$ is a pretty helpful expression b/c it offers us a way to deal with smeared out charges. (i.e., when we have $\rho(\vec{r})$ instead of q_i)

Energy in continuous charge situations

$$W_{sys} = \frac{1}{2}\int dq\, \tilde{V}(\vec{r}) = \frac{1}{2}\int \tilde{V}(\vec{r}\,')\rho(\vec{r}\,')d\tau'$$

Here, $\tilde{V}(\vec{r})$ is the potential at point $\vec{r}$ due to all of ρ except right at $\vec{r}$, but this is irrelevant issue for ρ as there's no charge in an infinitesimal volume...

So the total energy of an electrostatic system is,

$$W_{sys} = \frac{1}{2}\int \rho(\vec{r})V(\vec{r})d\tau$$

But where is this energy stored?

* Spoiler alert! It's in the electric field!

- Let's see how that's the case,

With $\rho = +\varepsilon_0 \nabla\cdot\vec{E}$, $W_{sys} = \frac{1}{2}\int \rho(\vec{r})V(\vec{r})d\tau$ becomes,

$$W_{sys} = \frac{\varepsilon_0}{2}\int_{Vol} (\nabla\cdot\vec{E})V\,d\tau$$

We can integrate this using the 3D version of integration by parts,

$$W_{sys} = \frac{\varepsilon_0}{2}\left[\int_{Boundary} V\vec{E}\cdot d\vec{A} - \int_{volume} \vec{E}\cdot\nabla V\,d\tau\right]$$

If the volume is all space, then $V, \vec{E} \to 0$ far away

so as long as all the charges are localized (e.g., no good for the infinite sheet)

$$W_{sys} = -\frac{\varepsilon_0}{2}\int_{vol} \vec{E}\cdot\nabla V\,d\tau$$

$$W_{sys} = -\frac{\varepsilon_0}{2}\int_{vol} \vec{E}\cdot\nabla V\, d\tau \qquad \text{But} \quad \vec{E} = -\nabla V \text{ so,}$$

$$\boxed{W_{sys} = +\frac{\varepsilon_0}{2}\int E^2 d\tau}$$

So, It's $\vec{E}$ that stores the energy!

$\frac{1}{2}\varepsilon_0 E^2$ gives the energy density $\left(\frac{\text{stored energy}}{m^3}\right)$

Clicker Questions: Pt. charges & Capacitor