Now that we have developed a sense of different forms of (temporary) magnetization, we will start to explore how to compute $\vec{A}$ and thus $\vec{B}$ for materials with different magnetizations, $\vec{M}$. We won't worry about what form of magnetism (para or dia), as we will instead just write down $\vec{M}$ in most cases.

- Our model for materials is that the magnetization results from lots of little current loops.
  - Remember that we focus on scales that includes several hundred or thousand atom → Classical E&M uses Lorentz Averaging!

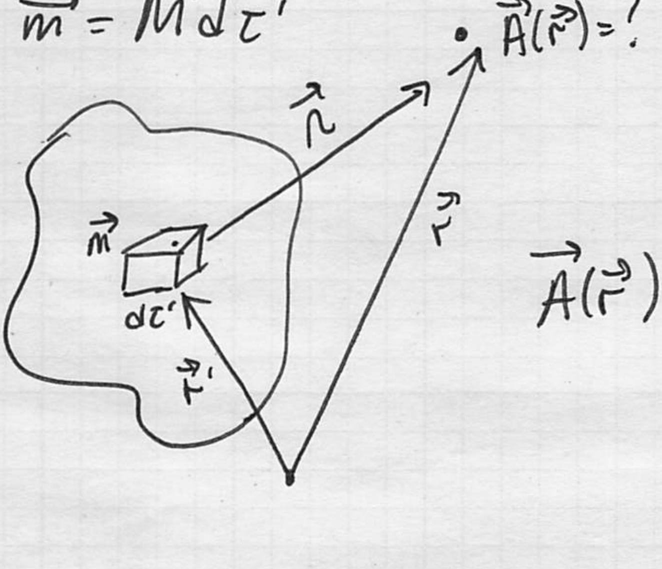
We will make heavy use of the vector potential here, $\vec{A}$.

$$\vec{A}_{\substack{\text{ideal}\\\text{dipole}}} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

In a material, we can consider a chunk of volume $d\tau'$, which has magnetization, $\vec{M}$.

in this model the dipole moment of that chunk is written thusly,

$$\vec{m} = \vec{M}\,d\tau'$$



We add up all the contributions to find $\vec{A}$,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{M}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

In principle we could just do this computation to find $\vec{A}(\vec{r})$, but this integral can be rewritten as the sum of two contributions: one due to bound surface current & one due to bound volume currents. [The proof is in the book.]

→ This is similar to what we found regarding charge in materials, by the way.

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_S \frac{\vec{K}_B(\vec{r}')}{\scriptstyle\mathcal{R}}\,da' + \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}_B(\vec{r}')}{\scriptstyle\mathcal{R}}\,d\tau'$$

Frankly, these integrals are much simpler and more intutive than the previous one.

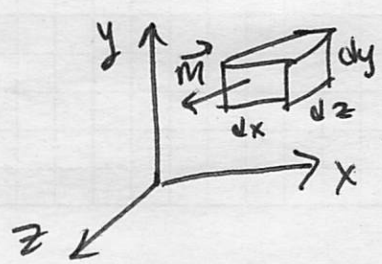
$\vec{A}$ arises from bound surface & volume currents that are given by the magnetization of the material. Namely,

$$\vec{K}_B = \vec{M}\times\hat{n} \quad \text{and} \quad \vec{J}_B = \nabla\times\vec{M}$$

[this is quite analogus to $\sigma_B = \vec{P}\cdot\hat{n}$ and $\rho_B = -\nabla\cdot\vec{P}$]

So there are these "effective" currents that are bound to the material, which give rise to $\vec{A}$ and thus $\vec{B}$. We can make sense of these two terms by considering a little chunk of material with

$$M = \frac{m}{dx\,dy\,dz}$$



Where does $\vec{K}_B$ come from?

So this little cube produces a magnetic dipole moment that can be modeled by a current running around 4 faces,

a little surface current K running around these 4 surfaces could give rise to $\vec{m}$.

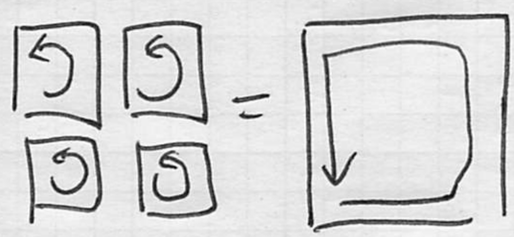
this little dipole is given by,

$m = I \times \text{area} = (K \cdot l_\perp)\,\text{area}$ $\qquad l_\perp = dz$ & area $= dx\,dy$

$m = K\,dx\,dy\,dz$ so that $M = K$

the direction of K is perpendicular to $\vec{m}$ + $\vec{M}$, and to the normal of each wall face; $\hat{M} \times \hat{n}$.

What's interesting is if $\vec{M}$ is constant then the the result arises from the outer surface because all chunks effectively cancel each other inside the material:



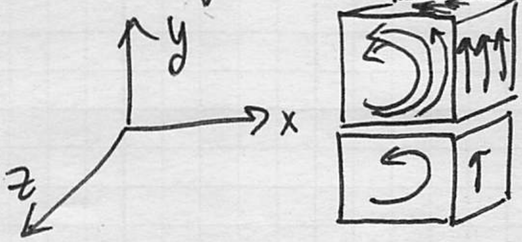
So a solid material with <u>uniform</u> tiny atomic currents can be modeled as simply having a macroscopic current running all around the surface of the material.

$$\vec{K}_B = \vec{M} \times \hat{n}$$

Where does $\vec{J}_B$ come from?

If the magnetization is not constant then we can suggest that some volume currents are at work.

Consider two cells as before, but now one has a larger magnetization, $\vec{M}$.



Here, the internal currents won't cancel anymore, so there's a net $\vec{J}$ inside (that can change w/ location)

In the case above,

$$I_{interior} = (J_B)_x \, dA_\perp = (J_B)_x \, dy\,dz$$

$I_{interior}$ is also given by the difference in upper and lower currents,

$$I_{interior} = I_{upper} - I_{lower} \Rightarrow$$

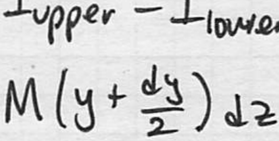
Recall, $M = K = I/dl_\perp = I/dz$
$I = M\,dz$

$$= M\left(y + \tfrac{dy}{2}\right)dz - M\left(y - \tfrac{dy}{2}\right)dz$$

$$= \frac{\partial M}{\partial y}\,dy\,dz = (J_B)_x\,dy\,dz$$

so that $$(J_B)_x = \frac{\partial M}{\partial y}$$ (for the case of $\vec{M} = M_z\hat{z}$.)

We could consider side by side cells:



and here we would get

$$(J_B)_y = -\partial M/\partial x$$

But

$$\nabla \times \vec{M} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & M \end{vmatrix} = \frac{\partial M}{\partial y}\hat{x} - \frac{\partial M}{\partial x}\hat{y}$$

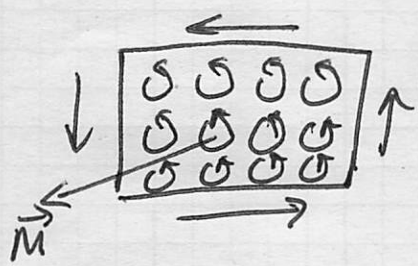
so, $$\vec{J}_B = \nabla \times \vec{M}.$$

Summary of Bound Current Results

Modeling matter as little current loops suggests two kinds of bound current:

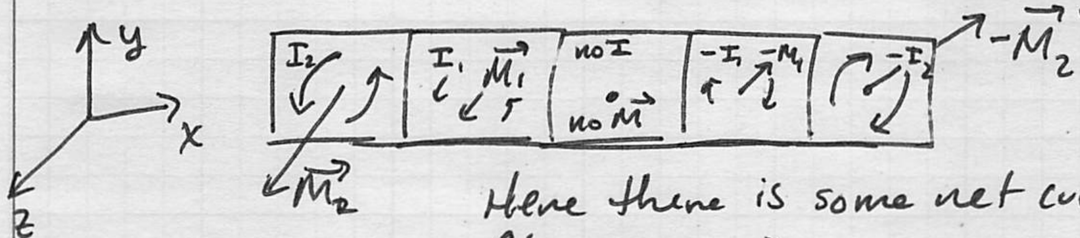
- Surface Current : $\vec{K}_B = \vec{M} \times \hat{n}$
- Volume Current : $\vec{J}_B = \nabla \times \vec{M}$

Surface Current arises from cancellations internal to the material



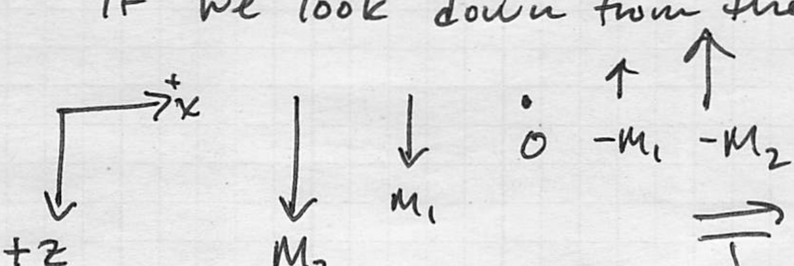
Some of the internal contributions cancel, which give rise to a net contribution on the surface of the cell. $\vec{K}_B = \vec{M} \times \hat{n}$

Volume Current arises when to contributions internal to the material don't exactly cancel.



Here there is some net current flowing inside.

If we look down from the +y direction

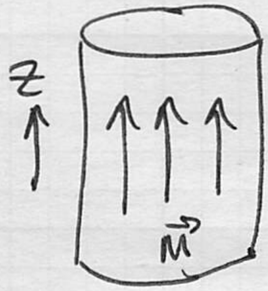


Definitely a curl to $\vec{M}$

$$\vec{J}_B = \nabla \times \vec{M}$$

$$A(\vec{r}) = \frac{\mu_0}{4\pi} \int_S \frac{\vec{K}_B(\vec{r}')}{\imath} da' + \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}')\, d\tau'}{\imath}$$

Example: Uniformly Magnetized Cylinder



Where are the Bound currents?

Assume $\vec{M} = M_0 \hat{z}$

It's uniform so $\nabla \times \vec{M} = 0$

— no volume bound currents.

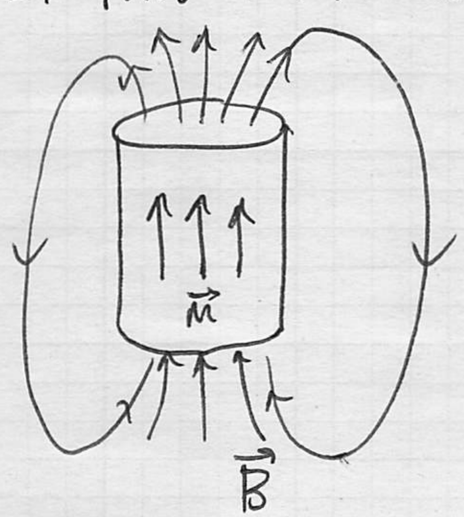
At the top and bottom the normal vector points in $\hat{z}$, so $\vec{M} \times \hat{n} = 0$ there.

What about around the cylindrical surface? there $\hat{n} = \hat{s}$ $\rightarrow$ the radially direction in cylindrical coordinates.

$$\vec{K} = \vec{M} \times \hat{n} = M_0 \hat{z} \times \hat{s} = M_0 \hat{\phi}$$

We have a uniform circulation of current!

$\Rightarrow$ This is like a finite solenoid.



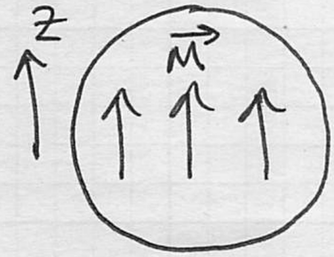
So it's a permanent magnet, which looks like a finite solenoid (inside & outside) even though there is no 'wire' with current wrapped around it.

The effect is a result of a bunch of atom contributions!

In principle we could compute the magnetic field by finding $\vec{A}(\vec{r})$ for this surface current,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_S \frac{\vec{K}}{\mathscr{r}} da'$$

Example: Uniformly magnetized sphere



Again, because it's uniform $\nabla \times \vec{M} = 0$ there are no bound volume currents.

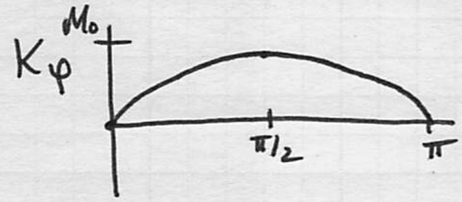
The surface currents are little more interesting because $\hat{n} = \hat{r}$; the usual radial direction in spherical coordinates.

Choose +z upward so,

$\vec{M} = M_0 \hat{z}$

$$\vec{K} = \vec{M} \times \hat{n} = M_0 \hat{z} \times \hat{r} = M_0 \sin\theta \, \hat{\varphi}$$

So it's a current that circulates around the sphere but whose strength varies as sine of the polar angle.



This is exactly a problem in Griffith's (Example 5.11)

$$\vec{B}_{inside} = \frac{2}{3} \mu_0 R \vec{M} \quad \underline{\underline{\text{uniform}}}$$

outside it is a perfect dipole field with dipole moment,

$$\vec{m} = \frac{4}{3} \pi R^3 \vec{M}$$