As we said, we can bring to bear all the mathematical tools we learned in vacuum. Let's do an example using the general ~~sto~~ solution to Laplace's equation is spherical coordinates (with azimuthal symmetry).

In a linear dielectric the bound charge is proportional to the free charge such that,

$$\rho_B = -\nabla\cdot\vec{P} = -\nabla\cdot\left(\varepsilon_0 \frac{\chi_e}{\epsilon}\vec{D}\right) = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f$$

If there's no free charge in the material, then there's no bound volume charge, all the net charge resides at the surface. thus, $\nabla^2 V = 0$ describes the potential. Our Boundary conditions are a bit different though,

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

$$\hookrightarrow \varepsilon_{above} E^{\perp}_{above} - \varepsilon_{below} E^{\perp}_{below} = \sigma_f$$

or

$$\epsilon_{above}\frac{\partial V_{above}}{\partial n} - \epsilon_{below}\frac{\partial V_{below}}{\partial n} = -\sigma_f$$

the potential is still continuous,

$$V_{above} = V_{below}$$

Example! A homogenous linear dielectric sphere (rad, R) with ϵ is placed in a uniform electric field $\vec{E} = E_0 \hat{z}$. Find $\vec{E}$ everywhere!

Laplace's equation applies because the dielectric is linear. We will use spherical coordinates for this, but first let's write out our Boundary Conditions!

$$V_{in} = V_{out} \quad \text{at } r = R$$

$$V_{out} \to -E_0 r \cos\theta \quad \text{for } r \gg R \quad \text{(as before)}$$

$$\epsilon \frac{\partial V_{in}}{\partial r} = \epsilon_0 \frac{\partial V_{out}}{\partial r}$$ why? b/c there's no free charge!

$$V(r,\theta) = \sum_l \left(A_l r^l + B_l / r^{l+1}\right) P_l(\cos\theta)$$

$r<R$:

All the B_l's $\to 0$ b/c must be finite! $V_{in} = \sum_l A_l r^l P_l(\cos\theta)$

$r>R$:

All the A_l's $\to 0$ b/c must be finite (except $l=1$) $\to$ b/c Boundary Condition

$$V_{out} = -E_0 r \cos\theta + \sum_l B_l / r^{l+1} P_l(\cos\theta)$$

Let's match V_{in} & V_{out} at $r=R$

$$\sum_l A_l R^l P_l(\cos\theta) = -E_0 R \cos\theta + \sum_l \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

for $l \neq 1$,

$$A_l R^l P_l(\cos\theta) = \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

so that $\quad A_l = \frac{B_l}{R^{2l+1}} \quad$ for $l \neq 1$

for $l = 1$,

$$A_1 R \cos\theta = -E_0 R\cos\theta + \frac{B_1}{R^2}\cos\theta$$

so that, $\quad A_1 = -E_0 + \frac{B_1}{R^3}$

Now let's match the normal derivatives,

$$\epsilon \frac{\partial V_{in}}{\partial r} = \epsilon_0 \frac{\partial V_{out}}{\partial r}$$

$$\epsilon \sum_l l A_l R^{l-1} P_l(\cos\theta) = -\epsilon_0 E_0 \cos\theta - \epsilon_0 \sum_l \frac{(l+1)B_l}{R^{l+2}} P_l(\cos\theta)$$

with $\epsilon/\epsilon_0 = \epsilon_R$,

$$\epsilon_R \sum_l l A_l R^{l-1} P_l(\cos\theta) = -E_0\cos\theta - \sum_l \frac{(l+1)B_l}{R^{l+2}} P_l(\cos\theta)$$

for $l \neq 1$,

$$\epsilon_R l A_l R^{l-1} P_l(\cos\theta) = -\frac{(l+1)B_l}{R^{l+2}} P_l(\cos\theta)$$

$$\epsilon_R l A_l R^{l-1} = -\frac{(l+1)B_l}{R^{l+2}}$$

$A_l = B_l = 0$ satisfies both this and the previous equation (if it works, it's unique!)

<u>for $l = 1$,</u>

$$\epsilon_R A_1 = -E_o - \frac{2B_1}{R^3}$$

so we have this ↑ and, $A_1 = -E_o + B_1/R^3$

Doing a little algebra,

$$\epsilon_R\left(-E_o + \frac{B_1}{R^3}\right) = -E_o - \frac{2B_1}{R^3}$$

$$B_1\left(\frac{\epsilon_R}{R^3} + \frac{2}{R^3}\right) = -E_o + \epsilon_R E_o$$

$$\frac{B_1}{R^3}(\epsilon_R + 2) = (\epsilon_R - 1)E_o$$

so that $B_1 = \left(\frac{\epsilon_R - 1}{\epsilon_R + 2}\right) R^3 E_o$

so that

$$A_1 = -\frac{3}{\epsilon_R + 2} E_o$$

thus,

$$V_{in}(r,\theta) = A_1 r\cos\theta = -\frac{3}{\epsilon_R + 2} E_o r\cos\theta$$

or

$$V_{in}(z) = -\frac{3}{\epsilon_R + 2} E_o z$$ thus,

$$\vec{E} = -\nabla V = \frac{3}{\epsilon_R + 2} E_o$$ it's uniform inside!

$$V_{out}(r,\theta) = -E_o r\cos\theta + \frac{B_1}{r^2} P_1(\cos\theta)$$

$$V_{out}(r,\theta) = -E_o r\cos\theta + \frac{(\epsilon_R - 1)/(\epsilon_R + 2)}{r^2} R^3 E_o \cos\theta$$

$$-\nabla V_{out} = -\frac{\partial V_{out}}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V_{out}}{\partial \theta}\hat{\theta} = \vec{E}$$

so

$$-\frac{\partial V_{out}}{\partial r} = E_0 - \frac{2(\epsilon_R - 1)/(\epsilon_R+2)}{r^3} R^3 E_0 \cos\theta$$

$$-\frac{1}{r}\frac{\partial V_{out}}{\partial \theta} = \frac{-1}{r}\left(E_0 r\sin\theta - \frac{(\epsilon_R-1)/(\epsilon_R+2)}{r^2} R^3 E_0 \sin\theta\right)$$

$$= +E_0\sin\theta + \frac{(\epsilon_R-1)/(\epsilon_R+2)}{r^3} R^3 E_0 \sin\theta$$

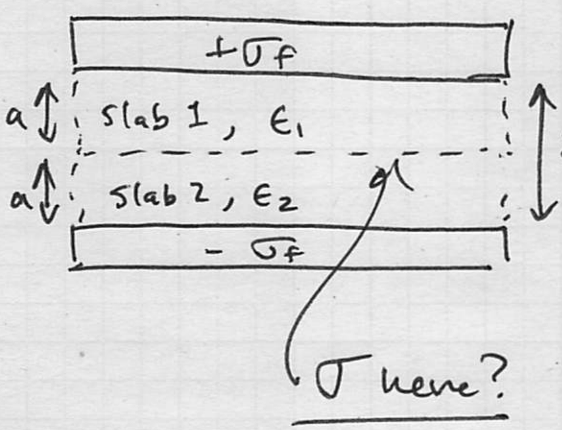
So,

$$\vec{E}_{out}(r,\theta) = E_0\left[\left(1 - 2\frac{(\epsilon_R-1)}{(\epsilon_R+2)}\left(\frac{R}{r}\right)^3\cos\theta\right)\hat{r} + \left(\sin\theta + \frac{(\epsilon_R-1)}{(\epsilon_R+2)}\left(\frac{R}{r}\right)^3\sin\theta\right)\hat{\theta}\right]$$

Why are these Boundary Conditions useful?
Let's see by example.

<u>Example</u>:

Consider a pair of linear dielectric slabs sitting between two capacitor plates with free charge, $+\sigma_f$ and $-\sigma_f$ on the top and bottom plates respectively. Each slab has a slightly different ϵ_R, ϵ_1 & ϵ_2 for the top and bottom slab, respectively.



We want to know the charge that builds up on the interface between the two slabs

We recall that $\vec{D}$ for this setup is simply,
$$\vec{D} = -\sigma_f \hat{z}$$
So that there's no free surface charge,
$$\vec{D}^{\perp}_{above} - \vec{D}^{\perp}_{below} = 0 = \sigma_f \quad \checkmark$$ As we expect.

So we need $\vec{E}$ to find the bound charge, it would actually gives us all the charge,
$$\vec{E}^{\perp}_{above}\cdot\hat{n} - \vec{E}^{\perp}_{below}\cdot\hat{n} = \sigma/\epsilon_0 \quad \text{where } \sigma = \sigma_f + \sigma_B$$

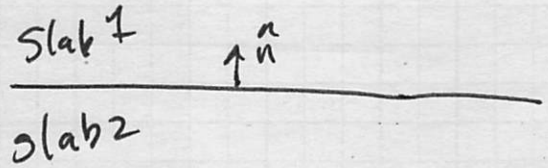
In slab 1,

$$\vec{E}_1 = \frac{\vec{D}}{\epsilon_R \epsilon_0} = \frac{\vec{D}}{\epsilon_1 \epsilon_0} = -\frac{\sigma_f}{\epsilon_1 \epsilon_0}\hat{z}$$

In slab 2,

$$\vec{E}_2 = \vec{D}/\epsilon_R\epsilon_0 = \frac{\vec{D}}{\epsilon_2 \epsilon_0} = -\frac{\sigma_f}{\epsilon_2 \epsilon_0}\hat{z}$$

To find the charge at that surface, we must choose an $\hat{n}$ for both $\vec{E}_1$ & $\vec{E}_2$, which is the same. in this case let's pick $+\hat{z}$,



$$\vec{E}^{\perp}_{above} \cdot \hat{n} - \vec{E}^{\perp}_{below} \cdot \hat{n} = \frac{\sigma}{\epsilon_0}$$

$$\vec{E}_1 \cdot \hat{n} - \vec{E}_2 \cdot \hat{n} = \sigma/\epsilon_0 \qquad \hat{n} = +\hat{z}$$

$$-\frac{\sigma_f}{\epsilon_1 \epsilon_0} + \frac{\sigma_f}{\epsilon_2 \epsilon_0} = \frac{\sigma_f}{\epsilon_0}\left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}\right) = \sigma/\epsilon_0$$

$$\sigma = \sigma_f\left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}\right) = \sigma_f\left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 \epsilon_2}\right)$$

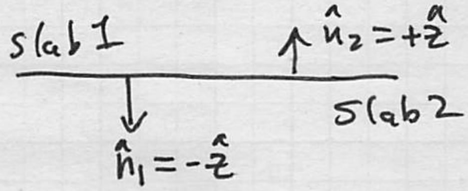
This method was fairly straight forward because of the Boundary Condition. We could have used the polarization instead and added the results from each $\vec{P}\cdot\hat{n}$ contribution. Let's do that,

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \implies \vec{P} = \vec{D} - \varepsilon_0 \vec{E}$$

$$\vec{P}_1 = -\sigma_f \hat{z} + \frac{\sigma_f}{\epsilon_1}\hat{z} = \sigma_f\left(\frac{1}{\epsilon_1} - 1\right)\hat{z}$$

$$\vec{P}_2 = -\sigma_f \hat{z} + \frac{\sigma_f}{\epsilon_2}\hat{z} = \sigma_f\left(\frac{1}{\epsilon_2} - 1\right)\hat{z}$$

At the surface, we can find the bound charge due to each polarization. Notice $\hat{n}$ is different for each.



$$\sigma_1 = \vec{P}_1 \cdot \hat{n}_1 = \sigma_f \left(\frac{1}{\epsilon_1} - 1\right) \hat{z} \cdot (-\hat{z}) = \left(1 - \frac{1}{\epsilon_1}\right) \sigma_f$$

$$\sigma_2 = \vec{P}_2 \cdot \hat{n}_2 = \sigma_f \left(\frac{1}{\epsilon_2} - 1\right) \hat{z} \cdot (+\hat{z}) = \left(\frac{1}{\epsilon_2} - 1\right) \sigma_f$$

$$\sigma_B = \sigma_1 + \sigma_2 = \left(1 - \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1\right) \sigma_f = \left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}\right) \sigma_f$$

$\sigma_B = \sigma$ b/c no σ_f so

$$\sigma = \left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 \epsilon_2}\right) \sigma_f$$ as we found using the BC's.

Notice that this method is a bit more difficult than the other method.