Clicker Question: $\Sigma$ E's?

Our interest is finding the real (total) electric field in and around a polarized piece of dielectric material. As you have seen, this polarization leads to bound charges in and on the material. Up to now this has been the story because our materials have been neutral (i.e., there's been not net charge applied to the material).

In general, these "free" charges - the ones we place in or on the material - contribute to the total electric field. Those free charges also further polarize the dielectric and add bound charges to the mix, which create different fields and superpose to give a different field.

The total charge density, $\rho$, is made up of those bound, $\rho_B$, and free charges, $\rho_f$.

$$\rho = \underset{\substack{\uparrow \\ \text{the "response"} \\ \text{of the dielectric}}}{\rho_B} + \underset{\substack{\uparrow \\ \text{the placed} \\ \text{charges}}}{\rho_f}$$

This $\rho$ is real and it creates the total, real electric field!

Gauss' Law is a known law of nature and applies to every situation (it's just not always useful). From Gauss', we find,

$$\nabla \cdot \vec{E} = \rho/\epsilon_0 = \frac{\rho_B + \rho_f}{\epsilon_0} = \frac{-\nabla \cdot \vec{P} + \rho_f}{\epsilon_0}$$

Here we've replaced $\rho_B$ with $-\nabla \cdot \vec{P}$ from our work with Polarization

$$\nabla\cdot\vec{E} = -\frac{\nabla\cdot\vec{P}}{\varepsilon_0} + \frac{\rho_f}{\varepsilon_0}$$ gives us,

$$\nabla\cdot\left(\vec{E} + \frac{\vec{P}}{\varepsilon_0}\right) = \frac{\rho_f}{\varepsilon_0}$$ such that,

$$\nabla\cdot\left(\varepsilon_0\vec{E} + \vec{P}\right) = \rho_f$$ we will define a new field, $\vec{D}$, that is the term in parentheses.

## The "Displacement" field or the "D-field"

$$\vec{D} \equiv \varepsilon_0\vec{E} + \vec{P}$$ the divergence of this field is equal to the

$$\nabla\cdot\vec{D} = \rho_f$$ $\Longleftarrow$ free charge density

By the divergence thereom we can develop an integral formalism of Gauss' Law for $\vec{D}$,

$$\int_V \nabla\cdot\vec{D}\, d\tau = \oint_S \vec{D}\cdot d\vec{A} = Q_{\text{free, enclosed}}.$$

Notice that $[D] = C/m^2$ not units of $E$!

## Why the D-field?

- $\rho_f$ are the "externally determined" charges – we placed those charges and they exist where we placed them.

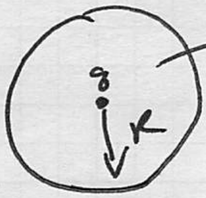
- $\rho_B$ are "self determined" charges; they respond to the ~~$\rho_f$~~ field from $\rho_f$ and any external fields. You don't choose $\rho_B$ and often don't know it!

If $\rho_f$ is sufficiently symmetric, we can apply our usual Gauss' Law method and "read off" $\vec{D}(r)$. If you know $\vec{P}$, then you can infer $\vec{E}$.

* $\vec{D}$ is a mathematical invention
  - a tool to help us find $\vec{E}$
  - it's often easier to find $\vec{D}$ first

Example: Charge embedded in rubber sphere.

A small (point) charge $+q$ is embedded in a rubber sphere (radius, $R$).




Find D everywhere.

The charge, $q$, creates an electric field $\vec{E}$ that polarizes the rubber, which in turn modifies $\rho$ in the sphere, altering $\vec{E}$ in some (as yet) unknown way. So $\vec{E}$ is not easy to find.

But this is where $\vec{D}$ can help us.

$$\oint \vec{D} \cdot d\vec{A} = Q_{\text{free, enclosed}}$$

this is just $q$! only free charge.

CQ: What's D? $\odot$

In the rubber, or outside, it makes no difference

there's no $1/\varepsilon_0$ here!

$$\oint \vec{D} \cdot d\vec{A} = q \qquad D 4\pi r^2 = q \qquad \vec{D}(\vec{r}) = \frac{q}{4\pi r^2}\hat{r}$$

Given $\vec{D}(\vec{r}) = \frac{q}{4\pi r^2}\hat{r}$, can we determine $\vec{E}$?

We defined $\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P}$ so,

$$\vec{E} = \frac{\vec{D}-\vec{P}}{\varepsilon_0} = \frac{\vec{D}}{\varepsilon_0} - \frac{\vec{P}}{\varepsilon_0}$$

→ Outside the sphere of rubber, there is no material, so $\vec{P}=0$ thus,

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0} - \cancel{\frac{\vec{P}}{\varepsilon_0}}^{\,0} = \frac{\vec{D}}{\varepsilon_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

Just the coulomb field – the rubber is polarized but neutral so it has no effect on the field outside the sphere.

→ Inside the sphere of rubber, $\vec{P} \neq 0$ so

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0} - \frac{\vec{P}}{\varepsilon_0}$$

If we don't know $\vec{P}$ we are still stuck!

(But we will learn that we can model many "normal" dielectrics as "linear", so we can figure of $\vec{P}$, given $\vec{E}$ (or $\vec{D}$))

<u>WARNING</u>: $\vec{D}$ is <u>not</u> "just like $\vec{E}$ but simpler!"

Given $\rho_{free}$, you can find $\vec{D}$ <u>if there's nice symmetry</u>

$$\oint \vec{D}\cdot d\vec{A} = Q_{free,\,enclosed}.$$

But if things are complicated then it's not possible (in general)

there is no $\vec{D} \neq \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_f\, d\tau\, \hat{\mathscr{r}}}{\mathscr{r}^2}$ b/c $\nabla\times\vec{D}\neq 0$! sometimes!

$$\nabla \times \vec{D} = \varepsilon_0 \underbrace{\cancel{\nabla \times \vec{E}}}_{0} + \nabla \times \vec{P}$$

- In vacuum, $\vec{P} = 0$ so you can easily find situations where $\oint \vec{P} \cdot d\vec{l} \neq 0$ thus $\nabla \times \vec{P} \neq 0$
- There is no potential for $\vec{D}$

$$V_D(\vec{r}) \neq \frac{1}{4\pi\varepsilon_0} \int_V \rho_f / \mathscr{r}^2 \, d\tau \quad \text{nope!}$$

and no Coulomb law either

$$\vec{D} \neq \frac{1}{4\pi\varepsilon_0} \int_V \rho_f \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau \quad \text{nope!}$$

Bottom Line:

- $\vec{D}$ is easy to compute (and thus useful) when you have a nice symmetry of $\rho_f$. (ala Gauss' Law)
- Otherwise, $\vec{D}$ might be helpful.

Boundary Conditions for $\vec{D}$

There is another aspect of $\vec{D}$ that can help us, which might be very useful with the linear dielectric story.

- You might recall that we found the normal component of the electric field was discontinuous by an amount $\sigma/\varepsilon_0$

$$\vec{E}^{above} \cdot \hat{n} - \vec{E}^{below} \cdot \hat{n} = \sigma/\varepsilon_0 \quad (\text{From Gauss})$$

- You might also recall that the parallel component of $\vec{E}$ is continuous

$$E_{\parallel}^{above} - E_{\parallel}^{below} = 0 \quad (\text{from } \nabla \times \vec{E} = 0)$$

Those were our usual boundary conditions on $\vec{E}$ and they are always true.

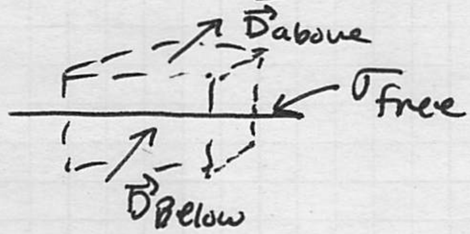
When we have a dielectric, it might be that we don't know $\sigma$. Some it may be known because its $\sigma_f$ (and we chose it), but some of it is $\sigma_B$, which is not known until we figure out $\vec{P}$!

So while those Boundary Conditions might be true, they might not be useful in figuring out $\vec{E}$.

Condition 1 on $\vec{D}$ $\Rightarrow$ $\vec{D}_\perp^{above} - \vec{D}_\perp^{below} = \sigma_{free}$

so $\oint \vec{D} \cdot d\vec{A} = \int \rho_{free}\, d\tau$ suggests

that $D_\perp^{above} - D_\perp^{Below} = \sigma_{free}$ ← which is most likely known.




So this condition might help us deduce $\vec{D}$, just like before.

Condition 2 on $\vec{D}$ $\Rightarrow$ $D_{\parallel}^{above} - D_{\parallel}^{below} = P_{\parallel}^{above} - P_{\parallel}^{below}$

$\nabla \times \vec{D} = \nabla \times \vec{P}$ suggests that

$\oint \vec{D} \cdot d\vec{l} = \oint \vec{P} \cdot d\vec{l}$ and says that

$P_{\parallel}^{above} - P_{\parallel}^{below} = D_{\parallel}^{above} - D_{\parallel}^{below}$

This might be helpful too if we know $\vec{P}$.

## Linear Dielectrics

A reasonable model ~~a~~ for many insulating materials is that of a linear relationship between the polarization of the dielectric and the total electric field.

$\vec{P} \propto \vec{E}$ this seems reasonable as $\vec{E}$ will "stretch" the dipoles.

(Recall: Griffiths suggested this with $\vec{p} = \alpha \vec{E}$)

- This model didn't have to be linear and it isn't always the case that $\vec{p} \propto \vec{E}$.

Take Note! $\vec{E}$ in this relationship it the total electric field (not just $\vec{E}_{ext}$!)

$\vec{E}_{ext}$ will polarize material, which superposes onto $\vec{E}_{ext}$. That is the polarization field also matters here. $\vec{E}_{tot} = \vec{E}_{ext} + \vec{E}_{polarization}$

So in this model, $\vec{P}$ is not necessarily given by $\vec{E}_{ext}$, it's proportional to the total resultant electric field, $\vec{E}_{tot}$ in the material.

## Model for Linear Dielectrics

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

Linear dielectrics are homogeneous & isotropic.

$\vec{P}$: polarization of the dielectric

$\chi_e$: unitless # "susceptibility"

$\vec{E}$: total $\vec{E}$ field

CQ: $\chi_e \to 0$ vs $\to \infty$?

$\chi_e \to 0$ vacuum — no polarization

$\chi_e \to \infty$ metal — very polarizable

Armed with,

$$\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P} \quad \text{and} \quad \vec{P} = \varepsilon_0 \chi_e \vec{E}$$

(in general) (for linear dielectrics)

we can construct alternative, useful forms for $\vec{E}$, $\vec{D}$, & $\vec{P}$ and their relationships.

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \quad \text{so,} \quad \vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$= \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E}$$

$$\vec{D} = \varepsilon_0 (1 + \chi_e) \vec{E}$$

so for linear dielectrics $\vec{D}$ is proportional to $\vec{E}$,

$$\vec{D} = \varepsilon \vec{E} \quad \text{where} \quad \varepsilon = \varepsilon_0 (1 + \chi_e) = \varepsilon_0 \varepsilon_R$$

permitivity of the dielectric

Dielectric Constant (unitless) [easy to measure & known for many different materials]

Now we see how useful $\vec{D}$ is now,

$$\oint \vec{D} \cdot d\vec{A} = Q_{\text{free, enclosed}} \Rightarrow \underline{\vec{E} = \vec{D}/\varepsilon}$$

Done!

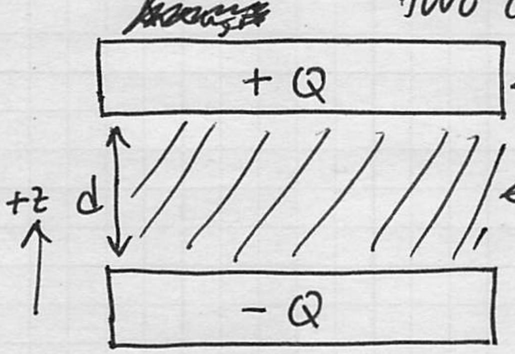
For linear dielectrics, if you know $\vec{D}$, $\vec{E}$, or $\vec{P}$ the other two are straight-forward to find if you know the dielectric constant for the medium.

$$\vec{D} = \varepsilon \vec{E} = (\varepsilon_0 \varepsilon_R) \vec{E} = \varepsilon_0 (1 + \chi_e) \vec{E}$$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \frac{\chi_e}{\varepsilon_R} \vec{D} = \left( \frac{\chi_e}{1 + \chi_e} \right) \vec{D}$$

Many forms of these expressions

Example : Insert a dielectric between two capacitor plates




Clicker Question: how to find $\vec{E}$?

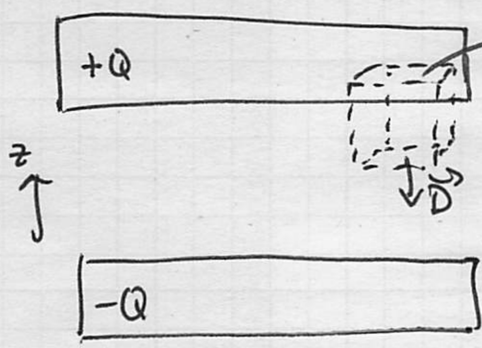
- If we want to find $\vec{E}_{tot}$ we run into issues because it's due to the free charge on the capacitor, which gives $\vec{E}_{ext} = -\sigma/\varepsilon_0 \hat{z} = \frac{-Q}{\varepsilon_0 A}\hat{z}$.
- But $\vec{E}_{ext}$ polarized the dielectric adding bound charges, which produce a polarization field that superposes with the external field to give the total field!

And $\vec{P} \neq \varepsilon_0 \chi_e \vec{E}_{ext}$ it's $\vec{P} = \varepsilon_0 \chi_e \vec{E}_{tot}$

and we don't know $\vec{E}_{tot}$ yet!

What about $\vec{D}$ instead? CQ: $\vec{D}$?

We can try $\oint \vec{D}\cdot d\vec{A} = Q_{free}$




CQ: $Q_{free}$? $\oint \vec{D}\cdot d\vec{A} = \sigma A$

Knowing $Q_{free}$ leaves us to figure of how $\vec{D}$ is related to $d\vec{A}$. We can assume D points the minus z direction in the dielectric.

CQ! $\vec{D}=0$ in metal? yup. it's. so, $\oint \vec{D}\cdot d\vec{A} = \sigma A$

CQ! What's $|\vec{D}|$? $D = \sigma$ ⟵ $= DA = \sigma A$

So we've found that,

$$\vec{D} = -\frac{Q}{A}\hat{z} \quad \text{throughout the dielectric!}$$

It's easy to find $\vec{D}$ when the symmetry is simple and we can use Gauss' Law.

Let's assume the dielectric is <u>linear</u> with dielectric constant, $\varepsilon_R$, then what's $\vec{E}_{tot}$?

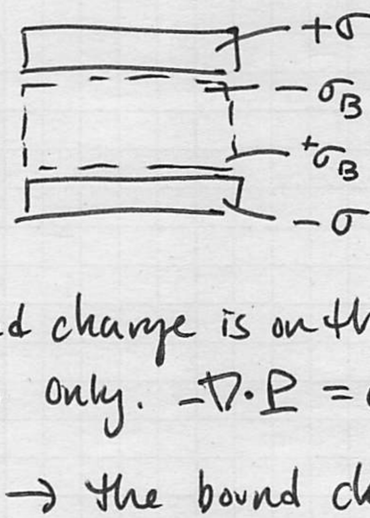
<u>Clicker Question</u>: $E$?

$$\vec{E}_{tot} = \vec{D}/\varepsilon_0\varepsilon_R \qquad \text{with } \vec{D} = -\frac{Q}{A}\hat{z}$$

we find, $\vec{E}_{tot} = \dfrac{-Q}{A\varepsilon_0\varepsilon_R}\hat{z}$ (It's $E_{ext}/\varepsilon_R$ in this case!)

– the dielectric caused the total field to be reduced by $1/\varepsilon_R$ (this is a common effect.)

<u>Why this reduction?</u>




Because it polarized! $\vec{P}\downarrow$

$$\vec{P} = \varepsilon_0\chi_e\vec{E}_{tot} = -\varepsilon_0\chi_e\frac{\sigma}{\varepsilon_0\varepsilon_R}\hat{z}$$

$$\vec{P} = -\frac{\chi_e}{1+\chi_e}\sigma\hat{z}$$

The bound charge is on the surfaces only. $-\nabla\cdot P = 0$

$$\vec{P}\cdot\hat{n} \begin{cases} -\dfrac{\chi_e}{1+\chi_e}\sigma & \text{top} \\ +\dfrac{\chi_e}{1+\chi_e}\sigma & \text{bottom} \end{cases} \Bigg\} \sigma_B$$

$\rightarrow$ the bound charges partially cancel $\sigma$ on the cap plates.

Effectively, $\sigma_{tot} = \sigma - \sigma_B = \sigma\left(1 - \dfrac{\chi_e}{1+\chi_e}\right)$

$$= \frac{\sigma}{1+\chi_e} = \frac{\sigma}{\varepsilon_R}$$

$E_{tot}$ arises from $\sigma_{TOT}$ suppressed by $1/\varepsilon_R$.

Finally note that $|\Delta V| = \left|\int \vec{E}\cdot d\vec{l}\right| = \frac{\sigma}{\varepsilon_0 \varepsilon_R} d$

So that $C = Q/\Delta V$ is $\varepsilon_R$ bigger!

This how we can increase capacitance!

Capacitors with dielectrics have:

- Bigger Capacitance
- Weaker $\vec{E}$ for a given $V \Rightarrow$ less likely to break down; spark.
- Can get $\varepsilon_R = 10000$ (Barium Titinate)

The stored energy is $\frac{1}{2} C \Delta V^2$ so this is larger by $\varepsilon_R$ also!

So Gauss' Law works but what about other methods? separation of variables (for example)?

In a linear dielectric $\rho_B = -\nabla\cdot\vec{P} \propto \nabla\cdot\vec{D} = \rho_{free}$

So with ~~...~~ no $\rho_f$'s then $\rho_B = \rho_f = 0$ and

$\nabla^2 V = 0$ so all this math can be used!