We've seen that Laplace's equation can describe a relatively simple situation – a capacitor – but it's very useful across many contexts.

Our approach to solving Laplace's equation needs to be generalized, so that we can tackle $\nabla^2 V = 0$ with given boundary conditions for <u>more</u> <u>general</u> <u>circumstances</u>.

It might be that $V(x,y,z)$ is quite complicated; in some cases, it might be $\propto \frac{1}{\sqrt{x^2+y^2+z^2}}$ but that only describes a very restricted set of cases.

In many cases we will find that

$V(x,y,z) =$ (function of x) * (another function of y) * (another function of z)

For example, $e^x \sin(y) \cos(z)$ but really need to know the <u>BC's</u>.

Even if this isn't the solution, we will find that it might be some combination (sum) of functions like this.

We will find using the <u>ansatz</u> (guess)

$$V(x,y,z) = X(x)\, Y(y)\, Z(z)$$ will solve $\nabla^2 V = 0$ in <u>many cases</u>.

We will often generate a general solution using this approach that needs to be matched to our boundary conditions — this will set the unknown constants and generate our particular solution.

- This technique is called,

Seperation of Variables

and is one very useful technique for solving partial differential equations.

- Basically, we try $V(x,y,z) \overset{?}{=} X(x)\,Y(y)\,Z(z)$

If it works w/ Boundary conditions, its our solution (uniqueness guarentees it!)

(if it fails, but a sum of functions works, then we are also good! Recall: superposition of V is ok! $V = V_1 + V_2 + V_3 + \ldots$ ✓)

So let's see how this goes with a solution in Cartesian coordinates

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Try $V = X(x)\,Y(y)\,Z(z)$ note $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{d^2 X(x)}{dx^2}\,Y(y)\,Z(z)$

partial derivative | total derivative | no x dependence so $\nabla^2 V = 0$ with $V = X(x)\,Y(y)\,Z(z)$ gives us,

$$X''(x)\,Y(y)\,Z(z) + Y''(y)\,X(x)\,Z(z) + Z''(z)\,X(x)\,Y(y) = 0$$

where X'' means $\frac{d^2X}{dx^2}$ we will divide both sides by $V = XYZ$ to get,

$$\underbrace{\frac{X''(x)}{X(x)}}_{\text{pure function of } x} + \underbrace{\frac{Y''(y)}{Y(y)}}_{\text{" " of } y} + \underbrace{\frac{Z''(z)}{Z(z)}}_{\text{" " of } z} = 0$$

* Clicker Question: $f(x) + g(y) + h(z) = 0$

With this test function, we find that each piece X''/X must be equal to a constant! so,

$$\frac{X''}{X} = C_1 \qquad \frac{Y''}{Y} = C_2 \qquad \frac{Z''}{Z} = C_3$$

with $C_1 + C_2 + C_3 = 0$

We reduced our problem to solving three simple 2^{nd} order ordinary differential equations!

Doesn't $X''(x) = C_1 X(x)$ look familar!?
It has very simple solutions.

$X''(x) = c_1 X(x)$ has the general solution,

$$X(x) = Ae^{\sqrt{c_1}\,x} + Be^{-\sqrt{c_1}\,x}$$

two undetermined constants (2nd order ODE)

*<u>Clicker Question</u>: what do solutions look like for $c_1 > 0$? $c_1 < 0$?

If $c_1 > 0$, $Ae^{\sqrt{c_1}\,x} + Be^{-\sqrt{c_1}\,x}$ exponential functions

If $c_1 < 0$, we get complex exponentials that can be written as sin & cos using the Euler theorem

$A'\sin(\sqrt{c_1}\,x) + B'\cos(\sqrt{c_1}\,x)$

If $c_1 = 0$ then we get a simple linear function

$A'' + B''x$

We changed our 3D PDE into 3 (easy) ODEs
But the cost we pay is lots of unknown constants showing up for which we use <u>Boundary</u> conditions to determine.

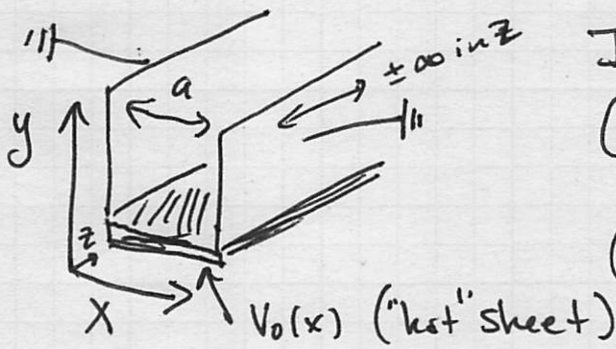
Let's start with a 2D Cartesian example to get a sense of the process we use to determine the coefficients.

So we will solve $\nabla^2 V = 0$ where we seek $V(x,y)$ (i.e., no z-dependence).

Physically, we setup something uniform in z so that nothing varies in that direction.

Example!

- Consider a metallic square gutter that extends in $+z$ forever.
- We will ground two sides of the gutter ($V=0$) and allow the bottom to have a voltage $V_0(x)$.
- The bottom is insulated from the side walls.



Inside the gutter:
$(0 < x < a)$
$(y > 0)$ is empty
(any z) $\nabla^2 V = 0$ there.

Boundary Conditions

$V(x=0, y>0) = 0$ grounded!

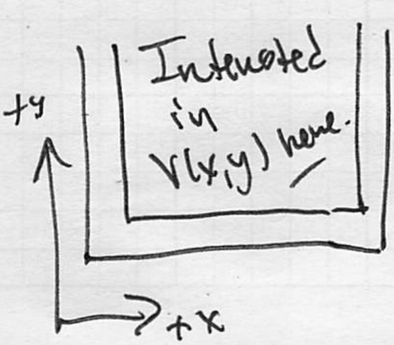
$V(x=a, y>0) = 0$ grounded!

$V(0<x<a, 0) = V_0(x)$ fixed voltage

$V(\text{any } x, y \to \infty) = 0$ physically reasonable b/c infinitely far from "hot" sheet.

We have no BCs for z, but no z-dependence uniform in z b/c ∞ extent.

Our problem is 2D! $V(x,y) = ?$



$\nabla^2 V = 0$

$V(x,y) = ?$ Let's try

$V(x,y) = X(x)\,Y(y)$!

Seperation of variables will give,

$X''(x) = c_1 X(x)$ and $Y''(y) = c_2 Y(y)$

with $c_1 + c_2 = 0$ so $c_2 = -c_1$!

Which one is positive? * Clicker Question

We can use the Physics at the boundary

If $c_1 > 0$ the $X = Ae^{+\sqrt{c_1}x} + Be^{-\sqrt{c_1}x}$,

which is no good b/c we will never have $X(0) = 0$ or $X(a) = 0$!

$c_1 = 0$ also no good b/c can't make it vanish at 0 & a unless its zero everywhere!

So $c_1 < 0$!

We will call $c_1 = -k^2$ to clearly indicate that it is a negative constant! so,

$c_2 = -c_1 = k^2$ is positive.

Thus,

$$\boxed{\begin{aligned} X(x) &= A\sin(kx) + B\cos(kx) \\ Y(y) &= Ce^{+ky} + De^{-ky} \end{aligned}}$$

is the general solution, but we have 4 coefficients and k to determine!

Lets use our Boundary conditions!

① $V(x=0)$ has to vanish as the left wall is grounded.

So $X(0) = 0$ which means,

$$X(0) = A\sin(0) + B\cos(0) = B = 0$$

thus,

$$X(x) = A\sin kx$$

② $V(y\to\infty)$ has to vanish as we are infinitely far from the "hot" sheet.

so, $Y(y\to\infty) = 0$ which means,

$$Y(y\to\infty) = Ce^{k(\infty)} + De^{-k(\infty)} \Rightarrow C = 0$$

$$Y(y) = De^{-ky}$$

So our solution thus far is,

$$V(x,y) = X(x)Y(y) = A\sin(kx)De^{-ky}$$

$$= C'\sin(kx)e^{-ky} \qquad C' = AD \text{ just a constant.}$$

③ $V(x=a) = 0$ as the right wall is grounded.

so,

$$C'\sin(ka)e^{-ky} = 0 \quad \text{for any } y.$$

$C' \neq 0$ b/c that makes $V(x,y) = 0$ for all x & y. (not true for $y=0$!)

So what do we do?

* Clicker Question: when does $\sin(ka)e^{-ky}$ vanish?

So, $\sin(ka) = 0$ so $k = n\pi/a$ will do the trick where n is any integer $(n>0)$ ($n=0$ is bad and $n<0$ is no different).

So we've got,

$$V(x,y) = C' e^{-ky} \sin(kx) \quad \text{with } k = \frac{n\pi}{a} \ (n = 1, 2, 3, \ldots)$$

[$n=0$ gives you $V=0$, which is no good
$n<0$ is the <u>same solution</u> with change of sign of C']

Our solution satisfies Laplace's equation and almost all of our boundary conditions except $V(x, y=0) = V_0(x)$.

- So if $V(x,0) = C' \sin(kx) = V_0(x)$ we are done! And we have $V(x,y)$ everywhere.
- But what if $V_0(x)$ is something else! We aren't stuck yet.

<u>Clicker Question</u>: $aV_1(r) + bV_2(r)$?

A linear combination of V_1 & V_2 if they both "work" ~~will~~ could be a solution as both satisfy Laplace,

$$\nabla^2 (aV_1 + bV_2) = \nabla^2 aV_1 + \nabla^2 bV_2 = 0 + 0 = 0$$

Turns out we have many solutions!
For any integer n,

$$V_n(x,y) = C_n \sin\left(\frac{n\pi x}{a}\right) e^{-n\pi y/a}$$

Satifies $\nabla^2 V = 0$ & 3 of our BCs!

so,

$$V(x,y) = \sum_n V_n(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) e^{-n\pi y/a}$$

also satisfies $\nabla^2 V = 0$ & 3 BC's

Can we pick our C_n's so that $V(x,0) = V_0(x)$?
- If we can, we got our solution!

$$V(x,0) = \sum_n C_n \sin\left(\frac{n\pi x}{a}\right) e^0 = \sum_n C_n \sin\left(\frac{n\pi x}{a}\right)$$

so, $\displaystyle\sum_n C_n \sin\left(\frac{n\pi x}{a}\right) = V_0(x)$ (Recall $V_0(x)$ would be a given function!)

It turns out there is an integral method called "Fourier's trick" that will get us what we need and let us find C_n for "well-behaved" $V_0(x)$'s.

Clicker Question! Remember $\displaystyle\int_0^{2\pi} \sin(2x)\sin(3x)\,dx$

Fourier's Trick

the functions $\sin\left(\frac{n\pi x}{a}\right)$ are a complete orthonormal set of functions

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n'\pi x}{a}\right) dx = \begin{cases} 0 & n \neq n' \\ a/2 & n = n' \end{cases}$$

Let's use this,

$$V_0(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)$$

multiply both sides by $\sin\left(\frac{n'\pi x}{a}\right)$ and integrate from $0 \to a$

$$\int_0^a \sin\left(\frac{n'\pi x}{a}\right) V_0(x)\,dx = \int_0^a \sum_n c_n \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n'\pi x}{a}\right)dx$$

$$\int_0^a V_0(x)\sin\left(\frac{n'\pi x}{a}\right) = \sum_n c_n \underbrace{\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n'\pi x}{a}\right)dx}_{\substack{0 \quad n\neq n' \\ \frac{a}{2} \quad n=n'}}$$

$$\int_0^a V_0(x)\sin\left(\frac{n'\pi x}{a}\right)dx = c_{n'}\frac{a}{2}$$ so,

$$\boxed{C_n = \frac{2}{a}\int_0^a V_0(x)\sin\left(\frac{n\pi x}{a}\right)dx}$$

gets us all the c_n's!

that will give us $V(x,y)$ everywhere!

$$V(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)e^{-n\pi y/a}$$

with $$C_n = \frac{2}{a}\int_0^a V_0(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

notice why $V_0(x)$ needs to be "well behaved"

Let's say $V_0(x) = V_0$ a constant potential

$$C_n = \frac{2}{a}\int_0^a V_0 \sin\left(\frac{n\pi x}{a}\right)dx = \frac{2V_0}{a}\int_0^a \sin\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2V_0}{a}\left[-\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\right]_0^a$$

$$C_n = -\frac{2V_0}{n\pi}\left(\cos(n\pi) - 1\right)$$

so,

$$C_1 = -\frac{2V_0}{\pi}(-2) = \frac{4V_0}{\pi} \quad ; \quad C_2 = -\frac{2V_0}{2\pi}(1-1) = 0$$

In general,

$$C_{n\,odd} = \frac{4V_0}{n\pi} \qquad C_{even} = 0$$

thus,

$$V(x,y) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4V_0}{n\pi}\sin\left(\frac{n\pi x}{a}\right)e^{-n\pi y/a}$$

It's ugly but it's exact! What does it look like?

C_n is different based on $V_0(x)$!