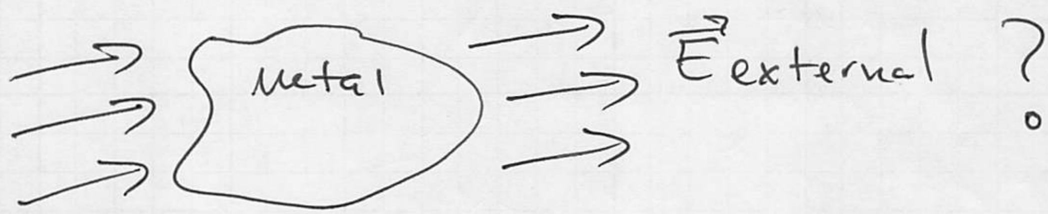
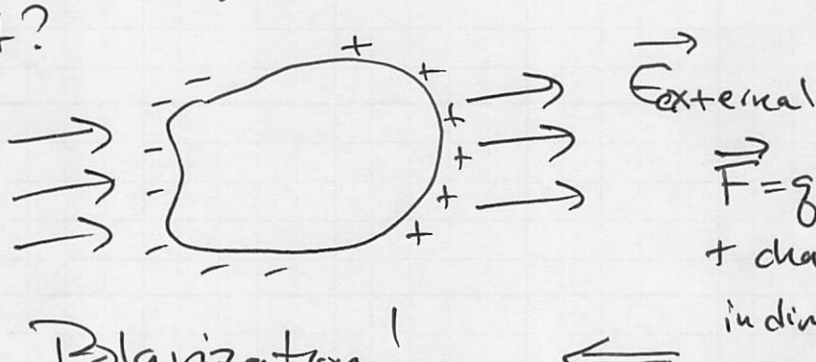
- We spent most of our time so far working with electric fields in space (i.e., where there are now objects ~~and~~ or materials). It's critical in the study of electrostatics to address how materials respond to electric fields as many applications of E&M leverage these interactions.

- We will restrict ourselves for now to metals or "conductors". The canonical problem we want to solve is this one:
  what happens when a conductor is placed in an electric field?



Consider a neutral conductor, what happens to the charges and how do we represent that?



$\vec{F} = q\vec{E}$ so
+ charges move in direction of $\vec{E}$,
− charges move opp direction of $\vec{E}$

Polarization! $\Longleftarrow$

The charges move until they feel no force $\Rightarrow$ <u>electrostatics</u>

For us, we will treat metals as <u>perfect</u> <u>conductors</u>, but know this is an approximation albeit a <u>very good one</u>.

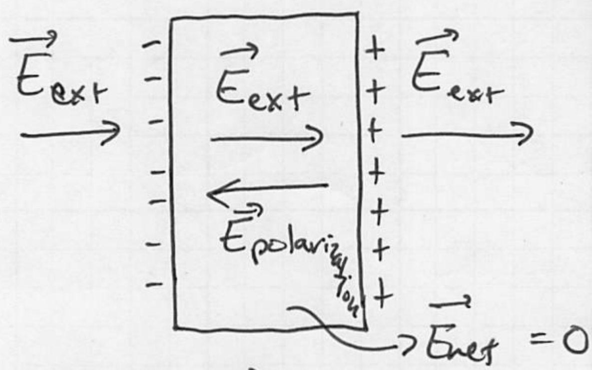
<u>Electron sea</u>

We think of metals as having ~~a~~ a <u>very mobile</u> sea of electrons that slosh around as needed in response to electric fields.

- From this conceptual model and the approximation of perfect conductance we find the following "axioms" or "consequences."

- Because charges are <u>free to move</u> inside, they can respond (instantly & without losses) to forces, which gives:

(1) the Electric field inside a metal is zero!
  - this is the net electric field due to all charges including those contained by the metal.



$\vec{E}_{polarization}$ is the field due to the rearranged charges on the metal

$\vec{E}_{polarization} = -\vec{E}_{metal}$ everywhere in the metal so that $\vec{E}_{net} = 0$.

(2) there is no excess charge in the bulk of the metal. Notice, we say, excess as the metal is made of atoms so there's obviously charges there but at the macro level we model it as no charge.

$*\rho = \varepsilon_0 \nabla \cdot \vec{E} = 0$ <u>inside</u>

* The argument that a metal has $\rho(\vec{r}) = 0$ inside is a complex and subtle one, which will discuss later.

(3) Excess charge must live on the outside edge of the metal. Think of our polarization example and also "consequence #2"

(4) Conductors are equipotential surfaces.

$$\Delta V(a \to b) = -\int_a^b \vec{E} \cdot d\vec{l} = 0 \quad \text{if } a, b$$

are both <u>in</u> or <u>on</u> the metal. b/c the field inside is <u>zero</u>.

(5) the electric field at the surface of the metal is perpendicular to the surface.

- If there were any parallel component of $\vec{E}$ then charges would move across the surface, which is <u>not electrostatic</u>!

* Clicker Questions: copper sphere

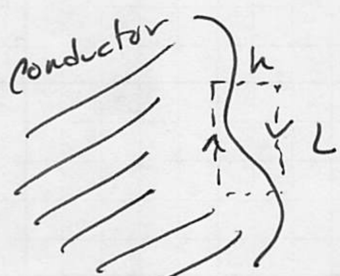
Formal Proof that $\vec{E}$ is $\perp$ to surface

$$\int_S \nabla \times \vec{E} \cdot d\vec{A} = \oint_C \vec{E} \cdot d\vec{l} = 0$$

From stoke's Law any circulation integral of $\vec{E}$ is zero (electrostatics)

$\nabla \times \vec{E} = 0$ in electrostatics

Let's apply this to a conductor at the surface



Consider $\oint \vec{E} \cdot d\vec{l}$ with $h \to 0$

This must be zero.

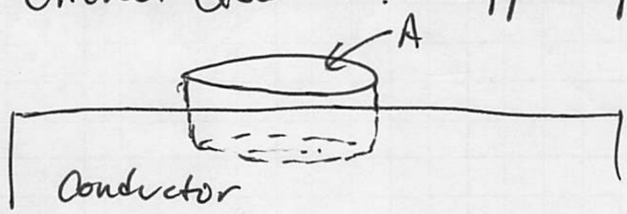
We will get,

$$0 \text{ inside} + \underset{h\to 0}{\text{tiny leg}} + E_{\parallel} \cdot L + \underset{h\to 0}{\text{tiny leg}} = 0$$

$E_{\parallel} = 0$ as all three terms are zero as $h \to 0$.

What is the value of that field outside the Conductor? $E_{\perp} = ?$

Clicker Question! Copper plate



$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

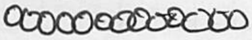
$$E_{out} A + E_{in} A + \underbrace{\text{walls}}_{0} = \frac{\sigma A}{\epsilon_0}$$

$E_{in} = 0$ it's a metal!!

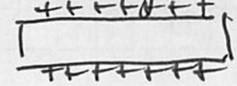
so, $E_{out} A = \sigma A / \epsilon_0$

$E_{out} = \sigma/\epsilon_0$ not $\sigma/2\epsilon_0$!

* It is not the case that $E_{out} = \sigma/2\epsilon_0$ for this conductor. The infinitely thin sheet of charge is very special!



vs.



A couple of consequences to consider with regard to conductors.

(1) Conductors polarize in the presence of external q's.

- They have to as $E_{inside} = 0$

(2) It makes it harder to solve for $V(\vec{r})$ & $\vec{E}(\vec{r})$ than in free space as we no longer know what ρ is "a-priori" as ρ will adjust!

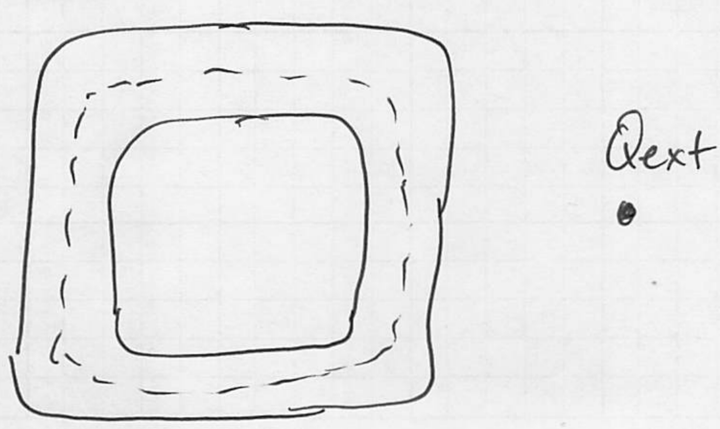
For Example $V = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho}{r}\, d\tau$ still holds

but for a pt. charge near a metal, what is ρ?

*Tutorial: Conceptually Understanding Conductors

Cavities & "Shielding"

Consider a situation where a conductor has a hole in the middle of it and some external point charge sits outside the metal.



We know $\vec{E} = 0$ in the metal, but what about in the hole?

From Gauss's Law $\oint \vec{E} \cdot d\vec{A} = Q_{enc}/\epsilon_0$,

we find that $Q_{enc} = 0$. <u>why</u>?

- the dashed line has $E=0$ everywhere
  thus, there's no flux $\Rightarrow Q_{enc} = 0$
  - Doesn't mean no charge, but no net charge!

★ If there's $+q$ in the hole, then $-q$ is distributed on the inside surface of conductor.

★ If there's no charge in the hole, then...

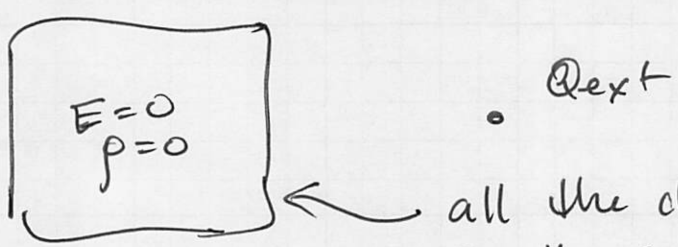
<u>$E=0$ in the hole</u>, <u>too</u>.

That last statement might be hard to accept. We know the metal polarizes, but how does it do so such that $E=0$ in the hole everywhere (and always!)?

We can try two arguments to make sense of this.

<u>Argument #1</u>:

Let's start with a solid conductor in the prescence of the external charge,



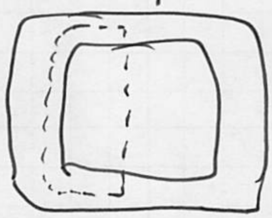
all the charges are on the surface, so when we cut the hole out from the inside we aren't removing any charges (that generate the polarization field)

Argument #2

We can go back to Stoke's again,

$$\oint \vec{E} \cdot d\vec{l} = 0 \quad b/c \quad \nabla \times \vec{E} = 0.$$

If $\vec{E} \neq 0$ then consider a loop that goes through the metal & the hole,



$\oint \vec{E} \cdot d\vec{l} = 0$

$E_{metal} = 0$ → only possible contribution is in the cavity (hole)

$\int \vec{E} \cdot d\vec{l} = 0$ for every line and either direction so $\vec{E}$ must be zero!

* We will learn the uniqueness theorem soon, which will tell us that when we find a solution, $V(\vec{r})$ or $\vec{E}(\vec{r})$ that satisfies your boundary conditions, there is no other solution!

$\vec{E} = 0$ is consistent and it's unique, so we found the solution to our problem.

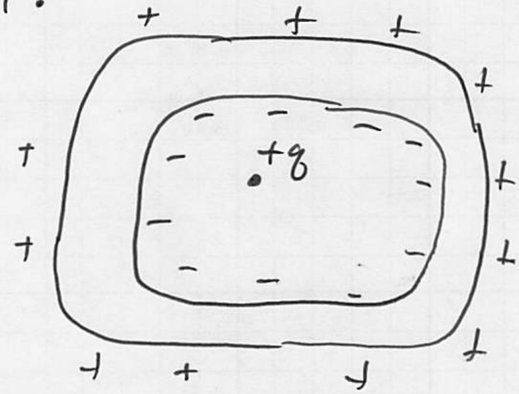
---

this is called "shielding" or the faraday cage effect. Inside a conductor $\vec{E} = 0$ (even in cavities, even with Q's outside, even if the conductor is charged!)

* shielding is a bad term – the fields are not blocked it is precisely because the exist in all space & superpose that we observe this effect for the net electric field.

* Clicker Question: what if there's a charge in the cavity?

If we put q in the cavity, we polarize the metal.



- You attract $-q$ to the inside wall and by charge conservation $+q$ ends up on the outside wall
- Notice that $E \neq 0$ in the cavity
- Also outside the field is nonzero because the total enclosed charge is now $+q$.

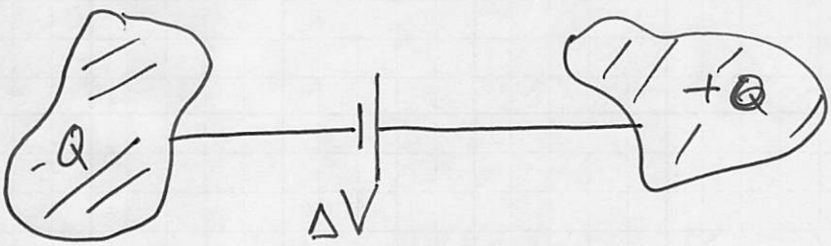
* Strangely, the field outside is the same as if we had the same conductor with no cavity, but a net charge $+q$ distributed over it's surface! (Uniqueness helps us here again.)

* Clicker Question: coax cable

Conductors can used store charge and energy to be used in other situations. One such way of configuring conductors to do this is to construct a capacitor.

The concept of a capacitor and capacitance is actually quite general and we will illustrate that first by conceptual example.

☆ Any pair of conductors will have a well defined potential difference, ΔV, because each one is an equipotential.



for these two conductors there are E-fields in all space so there's some $\Delta V = -\int_a^b \vec{E}\cdot d\vec{l}$

- so for each pair, amount of charge, and spatial configuration we can associate a potential difference ΔV.

- However, there is a quantity we can define that only depends on the spatial aspects of our problem (objects, configuration, & shape) that is, we can characterize the configuration without Q or ΔV!

Going back to ΔV & $\vec{E}$,

$$\vec{E} \propto Q \longrightarrow \vec{E} = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}^2}\hat{\mathscr{r}}$$

so is,

$$V \propto Q \longrightarrow V = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}}$$

Turns out that the ratio of Q to V is what we seek for the quantity called capacitance,

$C \equiv Q/\Delta V$ — it depends only on the objects, their shape, and configuration

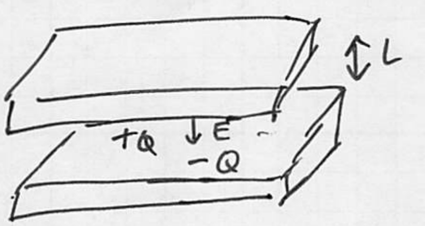
"it's purely a geometric quantity"

$$C = 1C/1V = 1 \text{ farad} = 1F$$

Examples:

Let's consider a set of metal plates w/ +Q and −Q on them.

★ Clicker Question: where does the charge live?



To find C, we need ΔV,

$\Delta V = -\int \vec{E}\cdot d\vec{l}$, so what's $\vec{E}$?

$\vec{E}$?

★ Clicker Question!

$\vec{E} = \frac{\sigma}{\epsilon_0}$ between plates, 0 elsewhere

(long, large plates
tiny gap)

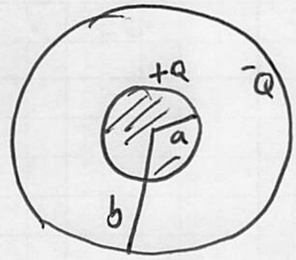
$$\Delta V = -\int \vec{E}\cdot d\vec{l} = +\frac{\sigma}{\epsilon_0}L$$

(integrate from −Q to +Q)

$$C = \frac{Q}{\Delta V} = \frac{Q}{\sigma L/\epsilon_0} = \frac{\epsilon_0 Q}{\sigma L} = \frac{\epsilon_0 Q}{Q/A\,L} = \epsilon_0 A/L$$

Another Example:

Consider a sphere radius a with charge +Q enclosed by spherical shell radius b > a with charge −Q



Here $\vec{E}$ in the space between is just a pt. charge.
- It's zero inside the metals and outside the shell!

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

$$\Delta V = -\int_a^b \vec{E}\cdot d\vec{l} = \frac{-1}{4\pi\epsilon_0}\frac{Q}{r}\Big|_a^b = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$C \equiv \frac{Q}{V} = 4\pi\epsilon_0\left(\frac{1}{\frac{1}{a}-\frac{1}{b}}\right) = 4\pi\epsilon_0\left(\frac{ab}{b-a}\right)$$

it's possible to define C for a single conductor (Assume: spherical shell @ ∞ with $-q$ on it)

for this configuration, $b \to \infty$ gives the result for just the sphere,

$$C = 4\pi\epsilon_0\left(\frac{1}{1/a - 1/b}\right)_{b\to\infty} \Rightarrow 4\pi\epsilon_0 a$$

Energy / Charge Storage

It's very common in applications to use capacitors to store energy / charge for use later. How do we determine the amount of stored energy?

We could compute / think about $\frac{\epsilon_0}{2}\int \vec{E}^2 d\tau$ or...

We could ask "how much work is needed" to charge up to Q?

The work to move dq over in the prescence of a known potential difference, ΔV, is,

$$dW_{\substack{\text{to move}\\ dq \text{ over}}} = dq \cdot \Delta V$$

[Note: $\Delta V = Q/C$ and thus depends on charge already present!]

$$W_{total} = \int_{q=0}^{q=Q} \Delta V \, dq = \int_0^Q \frac{q}{C} \, dq = \frac{1}{2} Q^2/C$$

So the total energy stored on our sphere

is $\frac{1}{2} \frac{Q^2}{4\pi\epsilon_0 a}$.

Clicker Question: capacitors & energy