We have shown that

$$Y(\theta,\phi) = \Theta(\theta)\,\Phi(\phi) \quad \text{solves} \quad H|E\rangle = E|E\rangle$$

if

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}\, e^{im\phi}$$

$$\Theta_\ell^m(\theta) = \sqrt{\frac{(2\ell+1)}{2}\frac{(\ell-|m|)!}{(\ell+|m|)!}}\; P_\ell^m(\cos\theta)$$

Spherical Harmonics

$$\boxed{Y_\ell^m(\theta,\phi) = (-1)^{(m+|m|/2)}\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}} \times P_\ell^m(\cos\theta)\, e^{im\phi}}$$

$$\ell = 0, 1, 2, 3, \ldots \qquad m = -\ell, -\ell-1, \ldots, \ell-1, \ell$$

$$Y_\ell^{-m}(\theta,\phi) = (-1)^m\, Y_\ell^m(\theta,\phi)$$

position rep. of $|lm\rangle$ in spherical coords.

$$\underbrace{|lm\rangle}_{} \doteq \underbrace{Y_l^m(\theta,\phi)}_{\int d\theta \int d\phi \text{ etc.}}$$

In QM Book

l	m	Y_l^m
0	0	$Y_0^0 = 1/\sqrt{4\pi}$
1	0	$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta$
1	± 1	$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi}$

so on.

Properties of Y_l^m's

(1) orthogonal on unit sphere

$$\langle l_1 m_1 | l_2 m_2 \rangle = \delta_{l_1 l_2} \delta_{m_1 m_2}$$

$$\langle l_1 m_1 | l_2 m_2 \rangle = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \, Y_{l_1}^{m_1 *} Y_{l_2}^{m_2}$$

$\sin\theta d\theta d\phi = d\Omega$ solid angle

$$\int Y_{l_1}^{m_1 *} Y_{l_2}^{m_2} d\Omega = \delta_{l_1 l_2} \delta_{m_1 m_2}$$

② Complete Basis

Laplace series

$$\psi(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \underbrace{c_{lm}}_{\text{coeff}} \underbrace{Y_l^m(\theta,\phi)}_{\text{function}}$$

$$\langle lm|\psi\rangle = c_{lm} = \int_0^{2\pi} d\phi \int_0^{\pi} Y_l^{m*}(\theta,\phi) \, \underline{\psi(\theta,\phi)} \times \sin\theta d\theta$$

$$C_n = \int \underline{f(x)} \sin(nx) dx$$

(3) transform under parity $\vec{r} \to -\vec{r}$ depends on ℓ

$$Y_\ell^m(\pi-\theta, \phi+\pi) = (-1)^\ell Y_\ell^m(\theta,\phi)$$

(4) Eigenstates H_{sphere}, L^2, L_z

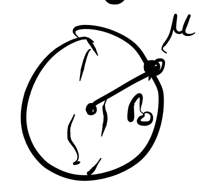
$$H_s Y_\ell^m = \frac{\hbar^2}{2I}\ell(\ell+1)\, Y_\ell^m$$

$$L^2 Y_\ell^m = \ell(\ell+1)\hbar^2\, Y_\ell^m$$

$$L_z Y_\ell^m = m\hbar\, Y_\ell^m$$

(5) Exhibit Degeneracy. $I = \mu r_0^2$

Energy $E = \frac{\hbar^2}{2I}\ell(\ell+1)$



$|\ell\ \ell\rangle, |\ell\ \ell-1\rangle, |\ell\ 0\rangle, \dots$

all have same energy Hsphere

$m = -l, -l+1, \ldots 0 \ldots, l-1, l$

$2l+1$ Energy degenerate

$\Rightarrow$ Sum over all deg. states

$L^2 = l(l+1)\hbar^2$ $\quad 2l+1$ deg.

$L_z = m\hbar$ $\quad l \geq m$ deg

Parity $\rightarrow$ sign change Y_l^m ?