$$\frac{-\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right.$$
$$\left.+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi+\underline{V(r)}\psi=E\psi$$

$$\boxed{H|E\rangle=E|E\rangle}$$

pos rep.
spherical
coords.

Particle on a ring.

fixed r_0, θ_0 $\rightarrow$ ϕ free

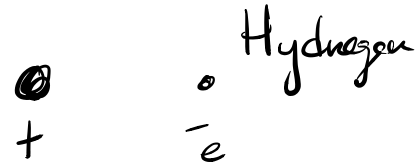
$$\frac{d^2\psi}{d\phi^2}=-B\psi$$

$$\psi=Ne^{+im\phi}\qquad m=0,\pm1,\pm2\ldots$$

$$1=\int_0^{2\pi}\psi^*\psi\,d\phi=N^2 2\pi\qquad N=\frac{1}{\sqrt{2\pi}}$$

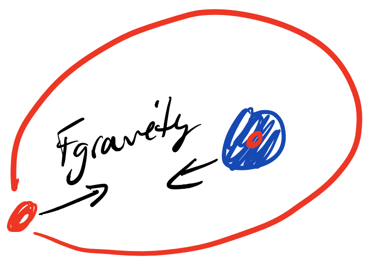
$$\boxed{|\psi\rangle\doteq\frac{1}{\sqrt{2\pi}}e^{im\phi}\qquad m=0,\pm1,\pm2\ldots}$$

$$\boxed{V(r) = \frac{e}{4\pi\varepsilon_0 r}}$$



Angular Momentum

Central potentials $\Longleftrightarrow$ Central forces

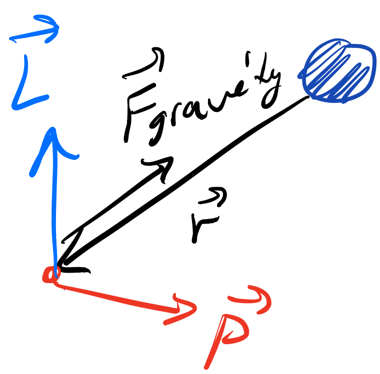


$$\vec{L} = \vec{r} \times \vec{p}$$

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{r} \times \vec{F}$$

$$= 0$$

$$\vec{r} \parallel \vec{F}$$



$$\frac{d\vec{L}}{dt} = 0$$

angular momentum conserved.

Emmy Noether

$\Rightarrow$ symmetry $\longleftrightarrow$ conservation law

Spherical sym $\rightarrow$ invariant to rotations

$\Downarrow$

ang mom. conserved

What about in quantum?

$$x \rightarrow \hat{x}$$
$$p \rightarrow \hat{p}$$
$\Big\}$ operators

$$L \rightarrow \hat{L} \; ?$$

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

$$\left.\begin{aligned} \hat{x} &= x \\ p_x &= -i\hbar \frac{d}{dx} \end{aligned}\right\} \text{pos rep.}$$

$$\hat{L}_z \doteq x\left(-i\hbar \frac{d}{dy}\right) - y\left(-i\hbar \frac{d}{dx}\right)$$

$$\boxed{\hat{L}_z \doteq -i\hbar\left(x\frac{d}{dy} - y\frac{d}{dx}\right)} \Leftarrow \text{pos rep}$$

$$L_z \doteq \left(+i\hbar \frac{d}{dp_x}\right)(p_y) - \left(+i\hbar \frac{d}{dp_y}\right)(p_x)$$

$$L_z \doteq +i\hbar\left(\frac{d}{dp_x} p_y - \frac{d}{dp_y} p_x\right) \quad \text{momentum rep}$$

$$x \rightarrow \hat{x} \qquad \hat{p} \rightarrow -i\hbar \frac{d}{dx} \quad ??$$

$$\left.\begin{aligned} [L_x, L_y] &= i\hbar L_z \\ [L_y, L_z] &= i\hbar L_x \\ [L_z, L_x] &= i\hbar L_y \end{aligned}\right\}$$

Spin relations

$$[S_x, S_y] = i\hbar S_z$$

$\vdots$

$$L^2 = \vec{L}\cdot\vec{L} = L_x^2 + L_y^2 + L_z^2 \Leftarrow$$

$$\left.\begin{array}{l}[L^2, L_x] = 0 \\ [L^2, L_y] = 0 \\ [L^2, L_z] = 0\end{array}\right\} \quad [S^2, S_x] = 0$$

like spin

Spin

eigenstate $|s\, m_s\rangle$

total spin — s; z proj. — m_s

$$S^2 |s m_s\rangle = s(s+1)\hbar^2 |s m_s\rangle$$

$$S_z |s m_s\rangle = m_s \hbar |s m_s\rangle$$

S_z basis

$|s\, m_s\rangle$ $\quad s = 0 \quad m_s = 0$

$s = 1 \quad m_s \to 1, 0, -1$

$m_s = -s, \ldots, +s$

$$|\psi\rangle = \frac{1}{\sqrt{2}} |1\, 1\rangle + \frac{1}{\sqrt{2}} |1\, {-1}\rangle$$

Same Eigenvalue Eqns
for ang. mom. states

$$\boxed{\begin{gathered} L^2 |\underbrace{\ell}_{\substack{\text{tot}\\ \text{ang.}\\ \text{mom.}}}\ \underbrace{m_\ell}_{\substack{z\\ \text{proj.}}}\rangle = \ell(\ell+1)\hbar^2 \underbrace{|\ell m_\ell\rangle}_{L_z \text{ basis}} \\ L_z |\ell m_\ell\rangle = m_\ell \hbar |\ell m_\ell\rangle \end{gathered}}$$

$\ell = 0, 1, 2, \ldots$
$m_\ell = -\ell, \ldots, 0, \ldots, +\ell$

eigenvalues whole integers
no half integers

Matrix formalism $\ell = 1$

$$L^2 \doteq 2\hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

same as S^2 for spin 1.

$$L_z \doteq \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Big\} \text{ Diagonal is } L_z \text{ basis}$$

evalues: $\hbar$, $0\hbar$, $-\hbar$

Change of coordinate sys. $x, y, z \downarrow r, \theta, \phi$

$$L_z = x\left(-i\hbar \frac{\partial}{\partial y}\right) - y\left(-i\hbar \frac{\partial}{\partial x}\right)$$

$$L_z \doteq -i\hbar \frac{d}{d\phi}$$

$$L^2 \doteq -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} \sin\theta \frac{d}{d\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$H|E\rangle = E|E\rangle$$

$$-\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \underbrace{\frac{1}{\hbar^2 r^2} L^2}_{L^2 \text{ operator}}\right)\psi + V\psi = E\psi$$

$$[H, L_z] = 0 \qquad [H, L^2] = 0$$

solutions will simultaneous eigenstates of H, L_z, L^2

$e^{iEt/\hbar}$ find time dep.

Example

$$|\psi\rangle = \frac{1}{\sqrt{3}}|11\rangle + \sqrt{\frac{2}{3}}|10\rangle$$

$l=1$ state superposition of 2 $m$es.

Probabilities of eigenvalues

$$P_{+\hbar} = 1/3 = |\langle 11|\psi\rangle|^2 \Leftarrow$$

$$P_0 = 2/3 = |\langle 10|\psi\rangle|^2 \Leftarrow$$

$$P_{-\hbar} = 0 = |\langle 1\,{-1}|\psi\rangle|^2 \Leftarrow$$

$$\langle L_z\rangle = \langle\psi|L_z|\psi\rangle \Leftarrow$$

$$= P_{+\hbar}(+\hbar) + P_0(0) + P_{-\hbar}(-\hbar)$$

$$= \frac{1}{3}(\hbar) + \frac{2}{3}(0) + 0(-\hbar) = \frac{\hbar}{3}$$

$$\langle L_x \rangle = ?$$

$$L_x \rightarrow P_{+\hbar}\ ?$$

$$\rightarrow |11\rangle_x = \frac{1}{2}|11\rangle_z + \frac{1}{\sqrt{2}}|10\rangle_z + \frac{1}{2}|1\,-1\rangle_z$$

$$L_x \rightarrow P_{+\hbar}\ ?$$

$$|{}_x\langle 11|\psi\rangle|^2$$

$$|\psi\rangle = \frac{1}{\sqrt{3}}|11\rangle + \sqrt{\frac{2}{3}}|10\rangle$$

$${}_x\langle 11|\psi\rangle = \frac{1}{2}\left(\frac{1}{\sqrt{3}}\right)\langle 11|11\rangle + \frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}}\langle 10|10\rangle$$

$$= \frac{1}{2\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{3}{2\sqrt{3}}$$

$$p_{th,Lx} = \frac{9}{4(3)} = \frac{9}{12} = \frac{3}{4}$$