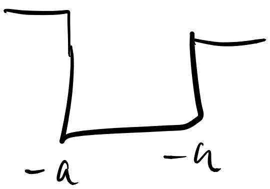
# Central Potentials & Eigenvalue Problem

1D models or abstract spin sys.




$\psi(x)$

$E_n$

position vs. momentum

(DiffyQ model)

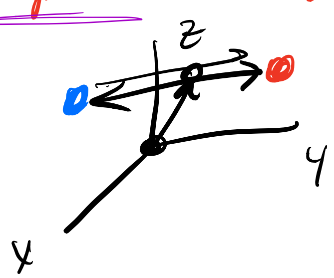
$|+\rangle$ $|-\rangle$

$|\psi\rangle$, $\langle S_z\rangle$, $|\psi(t)\rangle$

operator methods (linear algebra)

1. introduce 2 particles
2. let them interact $V(\vec{r}_1, \vec{r}_2)$
3. let them hang out in 3D

1. separate CM motion & motion relative to CM → 1 particle problem
2. central potential interaction




$V(|\vec{r}_1 - \vec{r}_2|)$

3. better specified in spherical coords

Hamiltonian for 2 particles that interact

$$H_{sys} = \frac{|p_1|^2}{2m_1} + \frac{|p_2|^2}{2m_2} + V(\vec{r}_1, \vec{r}_2)$$

$$\boxed{H_{sys} = \frac{|p_1|^2}{2m_1} + \frac{|p_2|^2}{2m_2} + V(\underline{|\vec{r}_1 - \vec{r}_2|})}$$

Simplification: CM & relative to CM

$$\vec{R}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \qquad \vec{r}_{rel} = \vec{r}_2 - \vec{r}_1$$

$$\vec{P}_{tot} = \vec{p}_1 + \vec{p}_2$$

$$\vec{P}_{rel} = \frac{m_1\vec{p}_2 - m_2\vec{p}_1}{m_1+m_2}$$

reduced mass (Classical mech, orbits?)

$$\frac{1}{\mu} \equiv \frac{1}{m_1} + \frac{1}{m_2} \qquad \mu = \frac{m_1 m_2}{m_1+m_2}$$

mu

$$\frac{\vec{p}_{rel}}{\mu} = \frac{\vec{p}_2}{m_2} - \frac{\vec{p}_1}{m_1} \Leftarrow$$

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Rewriting $H_{sys}$ interms CM & relative.

$$\boxed{H_{sys} = \frac{|\vec{P}_{tot}|^2}{2\underline{M}_{tot}} + \frac{|\vec{p}_{rel}|^2}{2\mu_{\leftarrow mu}} + V(\vec{r}_{rel})}$$

$$m_{nuc} \gg m_{electron}$$

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Motion relative to cm $\Rightarrow$ electron does this.

$$m_1 \sim m_2$$

$$H_{sys} = \boxed{H_{cm}} + \boxed{H_{rel}}$$

$$\boxed{\frac{|P_{tot}|^2}{2\underline{M}}} + \boxed{\frac{|P_{rel}|^2}{2\mu} + \underline{V(r_{rel})}}$$

## Energy Eigenvalue Eqn

$$H_{sys}\,\Psi(\vec{R}_{cm}, \vec{r}_{rel}) = E_{sys}\,\Psi(\vec{R}_{cm}, \vec{r}_{rel})$$

$$(H_{cm} + H_{rel})\,\Psi_{cm}(\vec{R}_{cm})\,\Psi_{rel}(\vec{r}_{rel})$$
$$= E_{sys}\,\Psi_{cm}(\vec{R}_{cm})\,\Psi_{rel}(\vec{r}_{rel})$$

$$\Psi(\vec{R}_{cm}, \vec{r}_{rel}) = \Psi_{cm}(\vec{R}_{cm})\,\Psi_{rel}(\vec{r}_{rel})$$

Beginning of Separation of Variables

Leap of faith: posit that each $\Psi$ satisifies its own eigenvalue eqn.

boring physics $\Rightarrow$ $H_{cm}\,\Psi_{cm}(\vec{R}_{cm}) = E_{cm}\,\Psi_{cm}(\vec{R}_{cm})$

interesting physics $\Rightarrow$ $H_{rel}\,\Psi_{rel}(\vec{r}_{rel}) = E_{rel}\,\Psi_{rel}(\vec{r}_{rel})$

$$E_{sys} = E_{cm} + E_{rel}$$

$$H_{cm} = \frac{|P_{tot}|^2}{2\underline{M}} \Leftarrow$$ free particle Ham. in 3D

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if $\vec{R} = \langle X, Y, Z \rangle$

$$\vec{P}_{tot} \doteq -i\hbar\left(\frac{d}{dX}\hat{i} + \frac{d}{dY}\hat{j} + \frac{d}{dZ}\hat{k}\right)$$

$$= -i\hbar \nabla_{R_{cm}}$$

$$-\frac{\hbar^2}{2m}\nabla_R^2 \psi_{cm} = E_{cm}\psi_{cm}$$

3D generalization of F.P.

$$\boxed{\psi_{cm} = \frac{1}{(2\pi\hbar)^{3/2}} e^{i(\underbrace{P_x X + P_y Y + P_z Z}_{\vec{P}_{tot}\cdot\vec{R}})/\hbar}}$$

$$\boxed{E_{cm} = \frac{1}{2M}(P_{tot})^2}$$

$$H_{rel} = \frac{|P_{rel}|^2}{2\mu} + V(r_{rel})$$

$$H = \frac{p^2}{2\mu} + V(r) \Leftarrow \text{relative frame.}$$

$$\vec{P}_{rel} = -i\hbar\left(\frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{k}\right) = -i\hbar\nabla_{rel}$$

$$H \doteq \frac{-\hbar^2}{2\mu}\nabla^2 + V(r)$$

$$H\psi = E\psi$$

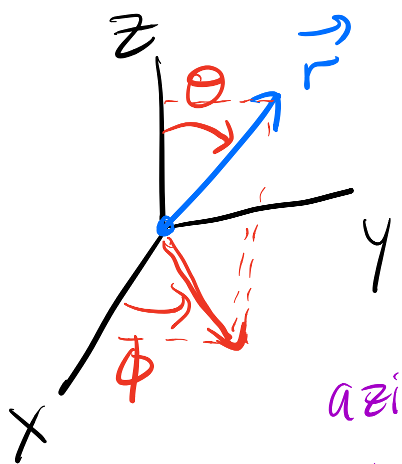
all variables relative

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\psi(\vec{r}) = E\psi(\vec{r})$$

all central potentials

# Central Potential, $V(r)$

$\Rightarrow$ use spherical coordinates
separate $r$ dep from $\theta, \phi$ dep.



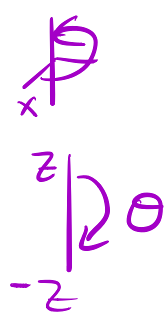

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

azimuthal $\phi \to 0 \to 2\pi$

polar $\theta \to 0 \to \pi$

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$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Laplacian in spherical

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)$$

$$|E\rangle \doteq \psi_E(r,\theta,\phi)$$

$$= \underbrace{R(r)}_{\text{radial}}\underbrace{Y(\theta,\phi)}_{\text{angular}}$$

next two weeks

know $V(r)$

angular momentum operators (analogy: spin)

$\frac{e}{4\pi\varepsilon_0 r}$ Coloumb

$\Rightarrow$ Hydrogen atom