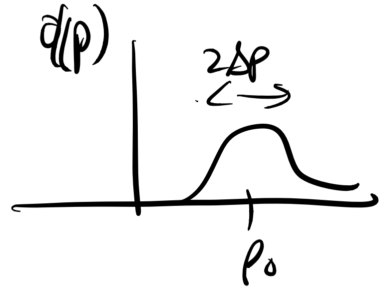
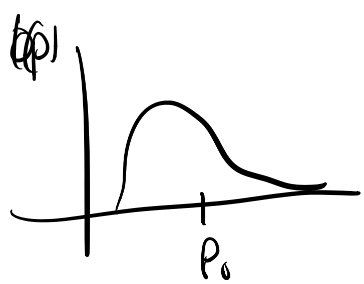
Wave packets & the Uncertainty Principle

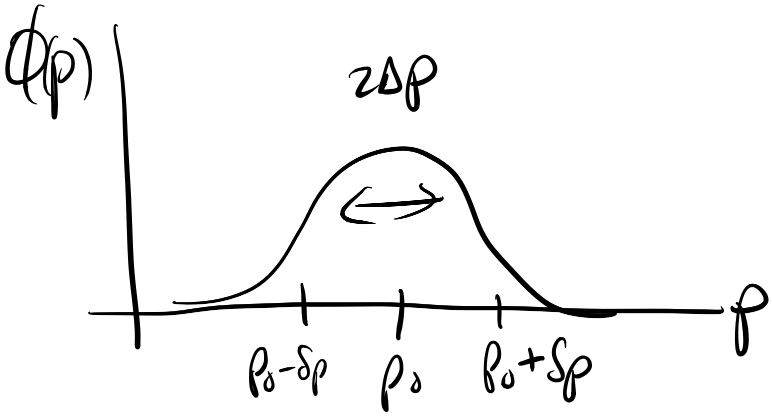
$\phi(p) \Leftarrow$ distribution of momentum for free particle

discrete

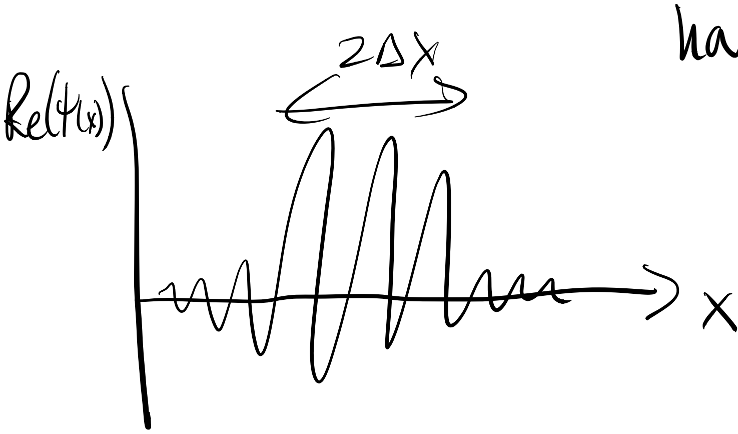




$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \phi(p)\, e^{ip\left(x - \frac{p}{2m}t\right)/\hbar}\, dp$$



$\Delta p \sim \delta p \implies \Delta p$ leads to having Δx



General Heisenberg U.P.

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle [A, B] \rangle \right|$$

$$\boxed{[\hat{x}, \hat{p}] = i\hbar} \Leftarrow$$

$$\boxed{\Delta x \Delta p \geq \frac{\hbar}{2}}$$

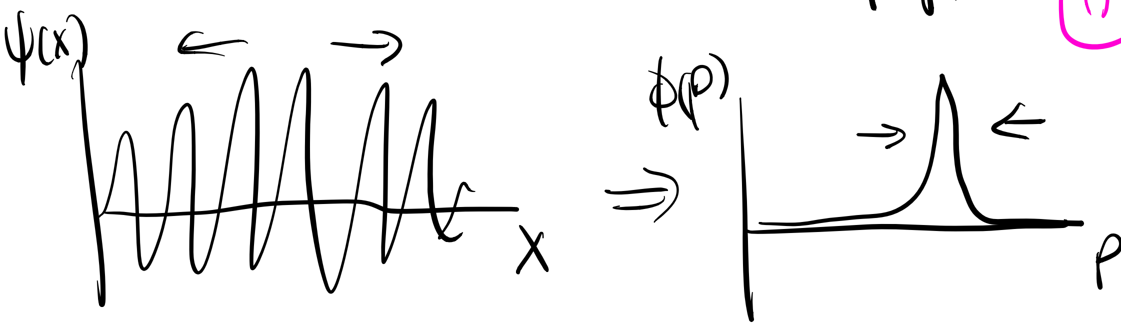
specific for $\Delta x \Delta p$

$$\Delta x \geq \frac{\hbar}{2} \frac{1}{\Delta p}$$

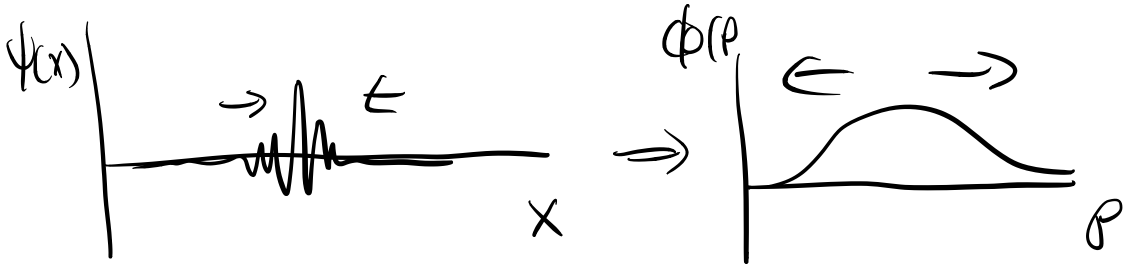
more $\Delta p \iff$ less Δx

less $\Delta x \iff$ more Δp

Broad $\psi(x) \implies$ narrow $\phi(p)$ (1)

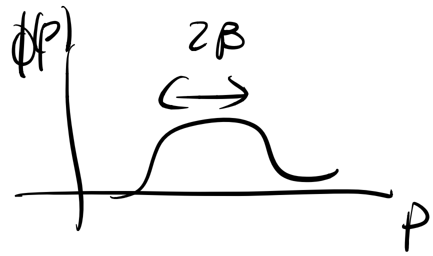


narrow $\psi(x) \implies$ broad $\phi(p)$



② how state evolves in time

Gaussian Beam



$$\Delta p = \beta$$

$$\Delta x = \frac{\hbar}{2\beta}\sqrt{1 + \left(\frac{2\beta^2 t}{m\hbar}\right)^2}$$ ⟸ Both from book

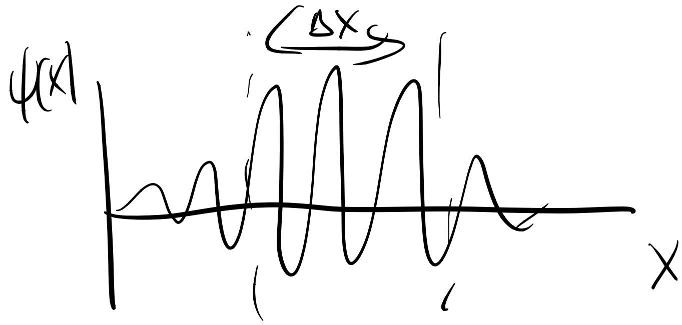
$$\Delta x \Delta p \geq \hbar/2$$

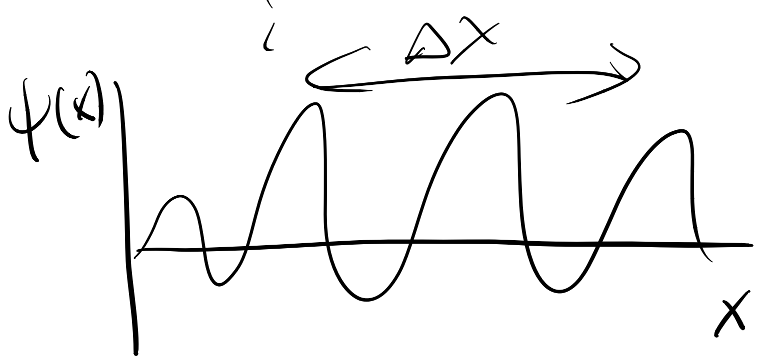
$$\Delta x \Delta p = \frac{\hbar}{2} \cdot f(t) \qquad f(t) > 0$$

minimum unc. state $t = 0$

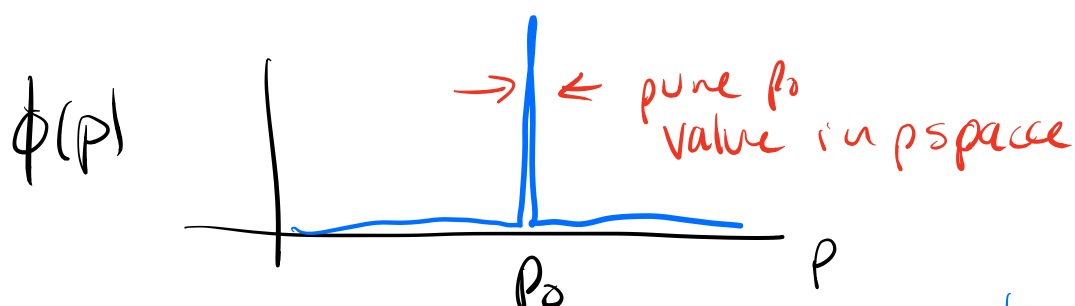
$\Delta x \Delta p = \hbar/2$ after that

$\Delta x \Delta p \geq \hbar/2$ ⟸ results from Δx getting larger.





Example 1: single p state



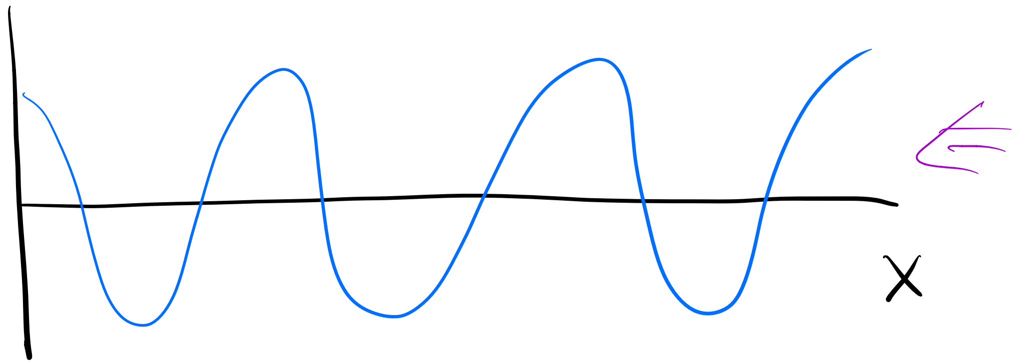
$|p_0\rangle$

Dirac Normalization

$$\langle x | p_0 \rangle = \psi_{p_0}(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ip_0 x/\hbar}$$

pure sinusoid

Ch 6.2/3?



infinite extent in x space

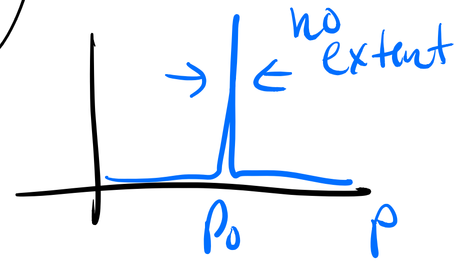
$$\phi_{p_0}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\varphi_{p_0}(x)\, e^{-ipx/\hbar}\, dx$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{ip_0x/\hbar}\, e^{-ipx/\hbar}\, dx$$

$$\boxed{\phi_p(p) = \underbrace{\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-i(p-p_0)x/\hbar}\, dx}_{\delta(p-p_0)}}$$

$$\phi_p(p) = \delta(p-p_0)$$

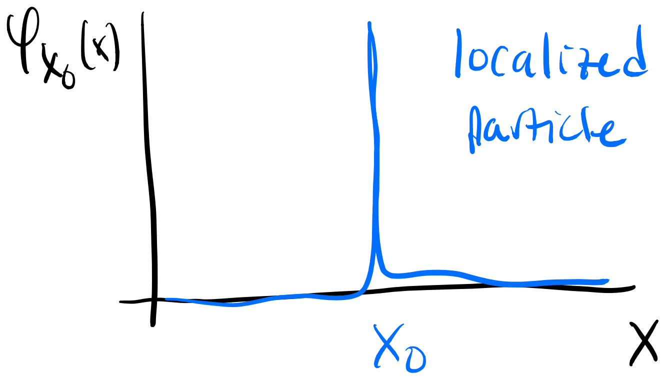
$$\underline{\langle p | p_0 \rangle} = \delta(p - p_0)$$



Infinite spatial extent $\Longleftrightarrow$ unique p_0

Example 2: Pure particle State

$$\langle x | x_0 \rangle = \varphi_{x_0}(x) = \underline{\delta(x - x_0)}$$



$$\hat{x} | x_0 \rangle = x_0 | x_0 \rangle$$

$$\hat{x}\, \delta(x - x_0) = x_0\, \delta(x - x_0)$$

$$\phi_{x_0}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \varphi_{x_0}(x)\, e^{-ipx/\hbar}\, dx$$

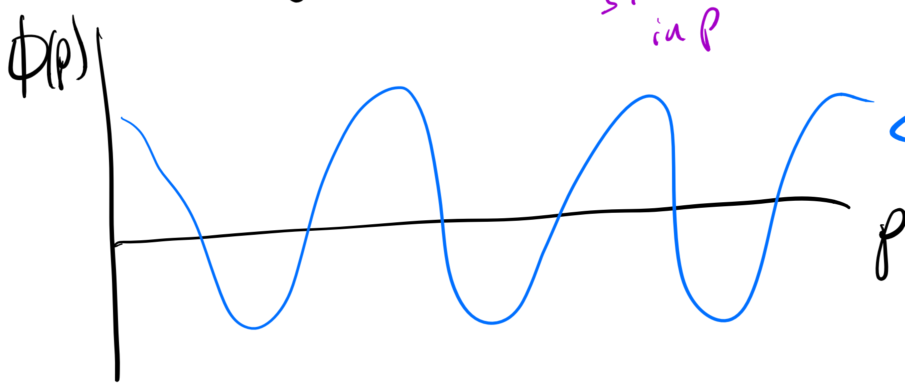
$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \delta(x - x_0)\, e^{-ipx/\hbar}\, dx$$

$$\int_{-\infty}^{\infty} \delta(x - x_0)\, f(x)\, dx = f(x_0)$$

$$\phi_{x_0}(p) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{-ipx_0/\hbar}$$

momentum space

sinusoidal in p



← infinite extent!

highly localized spatial extent ⟺ infinite momentum extent.

$$\boxed{\begin{aligned} \Delta x\, \Delta p &\geq \hbar/2 \\ \Delta x &\geq \hbar/2\, \frac{1}{\Delta p} \\ \Delta p &\geq \hbar/2\, \frac{1}{\Delta x} \end{aligned}}$$

Through lens

$\psi(x) \Leftrightarrow \phi(p)$

using F.T.

$$\boxed{\begin{aligned}\Delta x &= \sqrt{\langle x^2\rangle - \langle x\rangle^2}\\ \Delta p &= \sqrt{\langle p^2\rangle - \langle p\rangle^2}\end{aligned}}$$

$\langle p\rangle^2 = ?$ $\langle p^2\rangle = ?$

$\Delta p = ?$

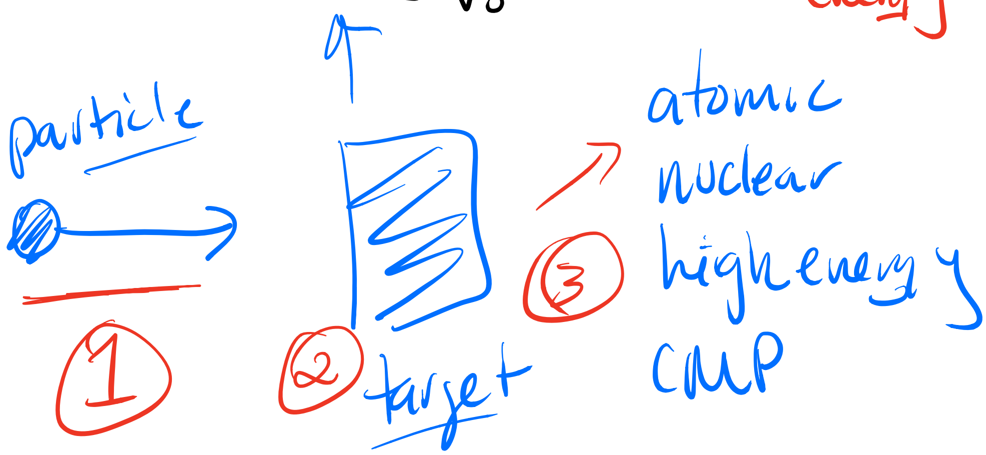
Free Particles in the presence of non zero V.

$E < V_0$ boundstates

$E > 0$ $V_0 = 0$ everywhere
free particle
Continuum of states

Now, $E > 0$ $V_0 < 0$ in some places

six coeffs & energy 7 unknowns

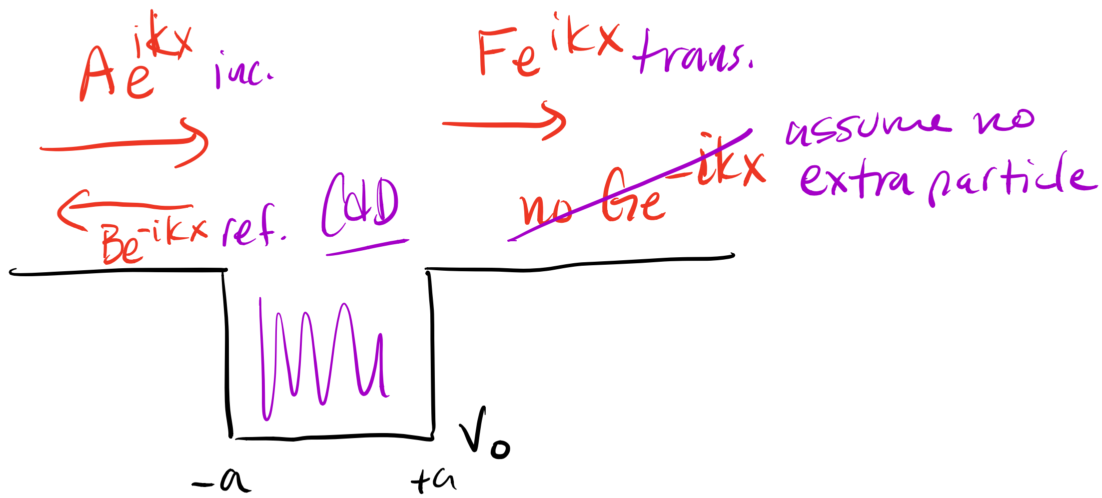


Scattering States

assume known $E \Leftarrow$ parameter

$\Rightarrow$ incoming W.F., scattered W.F., reflected W.F.

less concern about inside well.



$$V(x) = \begin{cases} 0 & x < -a \\ -V_0 & -a < x < a \\ 0 & x > a \end{cases}$$

$$\frac{d^2\psi}{dx^2} = \frac{-2mE}{\hbar^2}\psi_E \qquad \text{outside well}$$

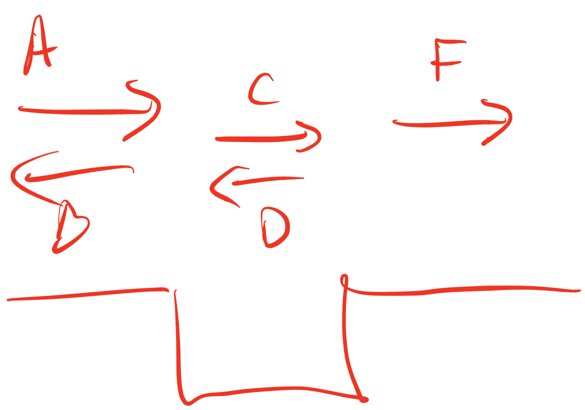
$$\frac{d^2\psi}{dx^2} = \frac{-2m(E-V_0)}{\hbar^2}\psi_E \qquad \text{inside}$$

$$k_2^2 = \frac{2mE}{\hbar^2} > 0 \qquad E > 0$$

$$k_1^2 = \frac{2m(E-V_0)}{\hbar^2} > 0 \qquad E - V_0 > 0$$

$$\frac{d^2\varphi}{dx^2} = -k_1^2 \varphi_E \quad \text{inside}$$

$$\frac{d^2\varphi}{dx^2} = -k_2^2 \varphi_E \quad \text{outside}$$



e^{ikx}

e^{-ikx}

(1) eliminate C&D for A, B, F (inc., ref., trans.)

(2) take A as given (luminosity beam)

(3) solve B/A & F/A

4

$$\frac{|B|^2}{|A|^2} = \text{Ref. coeff.} \qquad \frac{|F|^2}{|A|^2} = \text{Transmission coeff}$$