Free particle $V=0$

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi_E}{dx^2} = E\varphi_E \qquad k \equiv \sqrt{\frac{2mE}{\hbar^2}} > 0$$

$$\boxed{\frac{d^2\varphi_E}{dx^2} = -k^2\varphi_E}$$

$$\varphi_E = Ae^{ikx} + Be^{-ikx} \quad k \geq 0$$

$\Leftarrow$ energy eigenstates

$\Rightarrow$ let k run $[-\infty, +\infty]$

wave vector eigen states

$$\varphi_k(x) = Ae^{ikx} \qquad -\infty < k < \infty$$

Operate on $\varphi_k(x)$ with $\hat{p}$

$$\hat{p}\varphi_k(x) = \left(-i\hbar\frac{d}{dx}\right)\varphi_k(x)$$

$$= \left(-i\hbar\frac{d}{dx}\right)\left(Ae^{ikx}\right)$$

$$\boxed{\hat{p}\varphi_k(x) = \hbar k\,\varphi_k(x)}$$

$\hat{p}|p\rangle = p|p\rangle \qquad \hat{p}|k\rangle = \hbar k|k\rangle$

$p = \hbar k \qquad k = p/\hbar$

$$\boxed{\varphi_p(x) = A e^{ipx/\hbar}}$$

momentum eigenstate

$k = \frac{2\pi}{\lambda}$ ⇐ Wave Mech.

$p = \hbar k = \frac{h}{2\pi} k$ ⇐ free particle

$$\boxed{\lambda = h/p}$$

De Broglie

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momentum eigenstates are also energy eigenstates

$E_p = p^2/2m$

$$\psi_p(x,t) = \varphi_p(x)\, \underbrace{e^{-iE_p t/\hbar}}_{\text{usual phase}}$$

$$= A e^{ipx/\hbar} e^{-ip^2t/2m\hbar}$$

$$\boxed{\psi_p(x,t) = A e^{i\frac{p}{\hbar}\cdot\left(x - \frac{p}{2m}t\right)}}$$

$$f(x \pm vt)$$

position rep.

$V = P/2m$ ½ classical speed

phase velocity

$\Updownarrow$

momentum rep. $\phi_p(p)$

$A = \dfrac{1}{\sqrt{2\pi\hbar}}$ Dirac Normalization

Completeness : $\sum_i |a_i\rangle\langle a_i| = 1$

$\hookrightarrow \int_{-\infty}^{\infty} |p\rangle\langle p| \, dp = 1$

Change of basis,

$$\psi(x) = \langle x|\psi\rangle = \langle x|1|\psi\rangle$$

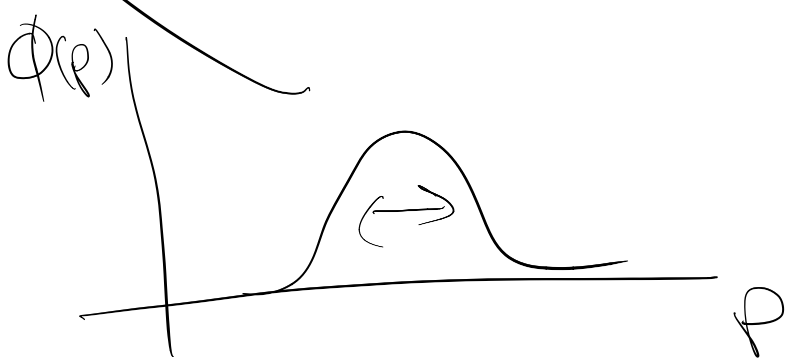
$$= \langle x|\left\{\int_{-\infty}^{\infty} |p\rangle\langle p|\,dp\right\}|\psi\rangle$$

$$= \int_{-\infty}^{\infty} \underbrace{\langle x|p\rangle}_{\substack{\text{proj. } |p\rangle \\ \text{in the } x \text{ basis} \\ \varphi_p(x)}} \underbrace{\langle p|\psi\rangle}_{\substack{\text{new. proj of } |\psi\rangle \\ \text{in } p \text{ basis} \\ \phi(p) \\ \text{momentum space wave fnc.}}}\,dp$$

$$\psi(x) = \int_{-\infty}^{\infty} \varphi_p(x)\,\phi(p)\,dp$$

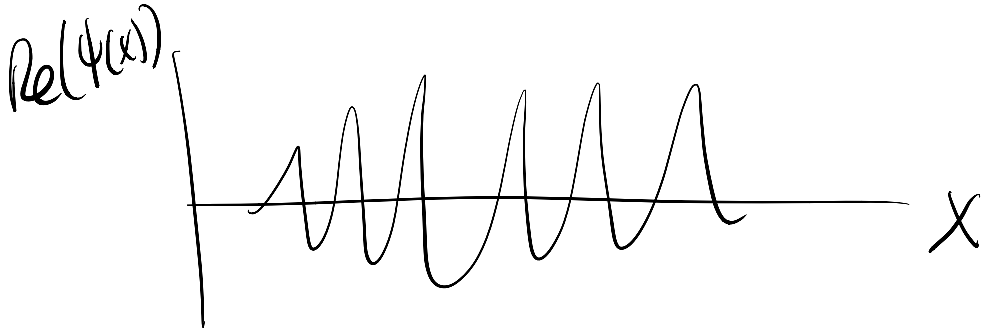
$$\Psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p)\, e^{ipx/\hbar}\, dp$$

Fourier transform of $\underline{\phi(p)}$



$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(x)\, e^{-ipx/\hbar}\, dx$$

inv. Fourier transform



$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$$\phi(p) = ?$$

$$\Rightarrow \phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \psi(x) e^{-ipx/\hbar}\, dx \quad \text{general}$$

$$\varphi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$$\varphi_n(x) = 0 \qquad x < 0$$

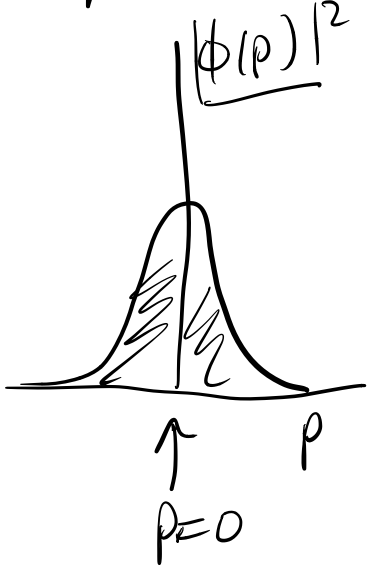
$$x > L$$

$$\Rightarrow \underbrace{\int_{-\infty}^{0}}_{0} + \underbrace{\int_{0}^{2}}_{\Uparrow} + \underbrace{\int_{L}^{\infty}}_{0}$$

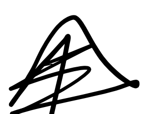
$\underline{\phi(p)}$ momentum space w.f.

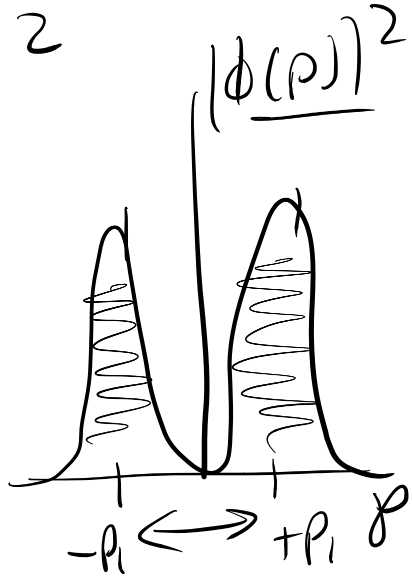
$\psi(x) \Leftrightarrow \phi(p) \qquad x \Longleftrightarrow p$

$n=1$ $\qquad$ $n=2$



$\rho(p)$





$\underline{\langle p \rangle = 0}$ $\leftarrow$ $\rightarrow \underline{\langle p \rangle = 0}$

$$P(x) = |\psi_n(x)|^2$$

$\langle x \rangle$ ✓ $\qquad \underline{\langle p \rangle}$

Finished Example

Starting from $\varphi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$

we can find $\phi(p)$ using the inv. F.T.

$$\phi_n(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \psi(x)\, e^{-ipx/\hbar}\, dx$$

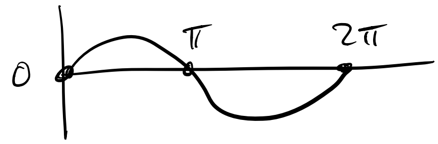
limits change b/c $\varphi_n = 0$ $x<0$ $x>L$

$$\phi_n(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) e^{-ipx/\hbar}\, dx$$

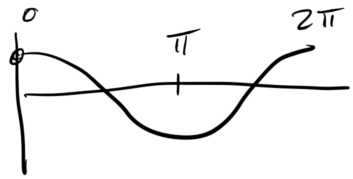
$$= \frac{1}{\sqrt{L\pi\hbar}} \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right) e^{-ipx/\hbar}\, dx}_{\text{Wolfram'ed}}$$

$$= \frac{1}{\sqrt{L\pi\hbar}} \left[\frac{\hbar L e^{-iLp/\hbar}\left(\pi\hbar n e^{iLp/\hbar} + \pi(-\hbar) n\cos(n\pi) - iLp\sin(n\pi)\right)}{\pi^2\hbar^2 n^2 - L^2p^2} \right]$$

$\sin(n\pi) = 0$



$\cos(n\pi) = (-1)^n$



$$\phi_n(p) = \frac{1}{\sqrt{L\pi\hbar}}\left[\frac{L\hbar\, e^{-iLp/\hbar}\left(n\pi\hbar\, e^{iLp/\hbar} - n\pi\hbar(-1)^n\right)}{n^2\pi^2\hbar^2 - L^2p^2}\right]$$

$$= \frac{1}{\sqrt{\pi\hbar L}}\left[\frac{n\pi\hbar^2L - (-1)^n n\pi\hbar^2 L\, e^{-iLp/\hbar}}{n^2\pi^2\hbar^2 - L^2p^2}\right]$$

$$\phi_n(p) = \frac{(n\pi\hbar^2L)\left(1-(-1)^n e^{-iLp/\hbar}\right)}{\sqrt{\pi\hbar L}\,(n^2\pi^2\hbar^2 - L^2p^2)}$$

$$\boxed{\phi_n(p) = \frac{1}{\sqrt{\pi\hbar L}}\left(\frac{n\pi\hbar^2}{L}\right)\left(\frac{(-1)^n e^{-iLp/\hbar} - 1}{p^2 - (n\pi\hbar/L)^2}\right)}$$

yuck! Looks like a mess

what about for $n=1$, $n=2$,

$$\phi_1(p) = \frac{1}{\sqrt{\pi\hbar L}}\left(\frac{\pi\hbar^2}{L}\right)\left(\frac{-e^{-iLp/\hbar}-1}{p^2-(\pi\hbar/L)^2}\right)$$

$$\rho_1(p) = |\phi_1(p)|^2$$

$$= \frac{1}{\pi\hbar L}\,\frac{\pi^2\hbar^4}{L^2}\,\frac{1}{(p^2-(\pi\hbar/L)^2)^2} \times$$

$$\left(-e^{-iLp/\hbar}-1\right)\left(-e^{+iLp/\hbar}-1\right)$$

$$= \frac{\pi\hbar^3}{L^3}\,\frac{1}{(p^2-(\pi\hbar/L)^2)^2}\times\left(1+1+e^{-iLp/\hbar}+e^{+iLp/\hbar}\right)$$

$$\rho_1(p) = \frac{\pi\hbar^3}{L^3}\,\frac{1}{(p^2-(\pi\hbar/L)^2)^2}\left(2+2\cos(Lp/\hbar)\right)$$