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Nearly every problem we have solved thus far has had an exact solution. However most systems require that we solve them approximately b/c typically they are too complex to yield exact analytical solutions.

The role of <u>Perturbation Theory</u> is to offer a framework for some class of these problems where the Hamiltonian can be separated into a part we can solve exactly and a part we believe has a small effect so that our solution can be approximated (typically analytically but not exclusively so).

We will introduce the general problem for a two state system and then

Show a specific example for Spin 1/2.

General Two Level Problem

2

For a general Two Level Problem our Hilbert space is 2D. We can formalize this by considering the energy eigenstates for the "original" or zeroth order Hamiltonian,

$$H = H_0 + H'$$

H': perturbation Hamiltonian,

H_0: original/zeroth order

Zeroth order

$$H_0 \doteq \begin{pmatrix} E_1^{(0)} & 0 \\ 0 & E_2^{(0)} \end{pmatrix} \qquad H_0|1\rangle = E_1^{(0)}|1\rangle \qquad H_0|2\rangle = E_2^{(0)}|2\rangle$$

perturbation

In the basis of the original Hamiltonian we consider a general perturbation,

$$H' \doteq \begin{pmatrix} H'_{11} & H'_{12} \\ H'_{21} & H'_{22} \end{pmatrix}$$

It is common to introduce a parameter λ that we later set to 1 to keep track of the order of corrections ($\lambda \to$ first order, $\lambda^2 \to$ secon order, etc.) 3

$$H = H_0 + \lambda H' \doteq \begin{pmatrix} E_1^{(0)} + \lambda H'_{11} & \lambda H'_{12} \\ \lambda H'_{21} & E_2^{(0)} + \lambda H'_{22} \end{pmatrix}$$

Note $H'_{12} = H'^{\dagger}_{21} \Rightarrow H$ must be Hermitian

$H \doteq \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}$ is the form of H

Can we find the exact eigenvalues?

$$H - IE \doteq \begin{pmatrix} a-E & c \\ c^* & b-E \end{pmatrix}$$

$$\det(H - IE) = 0$$

$$(a-E)(b-E) - cc^* = 0$$

$$E^2 - E(a+b) + ab - |c|^2 = 0$$

Quadratic Eqn.

$$E = \frac{+(a+b) \pm \sqrt{(a+b)^2 - 4(ab - |c|^2)}}{2}$$

$$= \frac{1}{2}(a+b) \pm \sqrt{\frac{1}{4}(a+b)^2 - ab + |c|^2}$$

$$\boxed{E = \frac{1}{2}(a+b) \pm \sqrt{\frac{1}{4}(a-b)^2 + |c|^2}}$$

Exact Solution

But we assume the contributions of H' are much smaller than H_0,
so that $(b-a) >> c$

Approximate Solutions?

$$E = \frac{1}{2}(a+b) \pm \frac{1}{2}(a-b)\left[1 + \frac{4|c|^2}{(a-b)^2}\right]^{1/2}$$

$$\boxed{E \simeq \frac{1}{2}(a+b) \pm \frac{1}{2}(a-b)\left[1 + \frac{2|c|^2}{(a-b)^2}\right]}$$

Approx Solutions

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$$E_1 \simeq \frac{1}{2}(a+b) + \frac{1}{2}(a-b)\left[1 + \frac{2|c|^2}{(a-b)^2}\right]$$

$$\boxed{E_1 \simeq a + \frac{|c|^2}{(a-b)}}$$

$$E_2 \simeq \frac{1}{2}(a+b) - \frac{1}{2}(a-b)\left[1 + \frac{2|c|^2}{(a-b)^2}\right]$$

$$\boxed{E_2 \simeq b - \frac{|c|^2}{(a-b)}}$$

Recall $\quad a = E_1^{(0)} + \lambda H_{11}' \qquad b = E_2^{(0)} + \lambda H_{22}'$

$\quad c = \lambda H_{12}' \qquad c^* = \lambda H_{21}'$

Translate this back to H_0 & H',

$$E_1 \simeq E_1^{(0)} + \lambda H_{11}' + \frac{\lambda^2 |H_{12}'|^2}{(E_1^{(0)} + \lambda H_{11}' - E_2^{(0)} - \lambda H_{22}')}$$

$$E_2 \simeq E_2^{(0)} + \lambda H_{22}' + \frac{\lambda^2 |H_{21}'|^2}{(E_1^{(0)} + \lambda H_{11}' - E_2^{(0)} - \lambda H_{22}')}$$

To second order in λ,

$$\boxed{\begin{aligned} E_1 &\simeq E_1^{(0)} + \lambda H_{11}' + \lambda^2 \frac{|H_{12}'|^2}{(E_1^{(0)} - E_2^{(0)})} \\ E_2 &\simeq E_2^{(0)} + \lambda H_{22}' + \lambda^2 \frac{|H_{21}'|^2}{(E_2^{(0)} - E_1^{(0)})} \end{aligned}}$$

We set $\lambda = 1$ and we can keep track of each contribution with $E_i^{(n)}$ where i is the state and (n) represents the correction order, ⑥

$$E_i \simeq E_i^{(0)} + E_i^{(1)} + E_i^{(2)}$$

Two state system

$E_i^{(0)} \Rightarrow$ unperturbed energy, H_{ii}

$E_i^{(1)} \Rightarrow$ first order correction, H'_{ii}

$E_i^{(2)} \Rightarrow$ second order correction, $\dfrac{|H_{ij}|^2}{(E_i^{(0)} - E_j^{(0)})}$