Ok we have finally finished! The 1

3D energy eigenstates of the Hydrogen atom are,

$$|n\ell m\rangle \doteq R_{n\ell}(r)\, Y_\ell^m(\theta,\phi)$$

Moreover this energy eigenstates are simultaneous eigenstates of L^2 & L_z!

$$\boxed{\begin{aligned} H|n\ell m\rangle &= -\frac{13.6\,eV}{n^2}|n\ell m\rangle \\ L^2|n\ell m\rangle &= \ell(\ell+1)\hbar^2|n\ell m\rangle \\ L_z|n\ell m\rangle &= m\hbar|n\ell m\rangle \end{aligned}}$$

The position representations of these eigenstates require knowing $n, \ell,$ & m. These states are tabulated in many QM books.

$$|100\rangle \doteq \psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

$$|200\rangle \doteq \psi_{200} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{2a_0}\right)^{3/2}\left[1-\frac{Zr}{a_0}\right] e^{-Zr/2a_0}$$

$$|210\rangle \doteq \psi_{210} = \frac{1}{2\sqrt{\pi}}\left(\frac{Z}{2a_0}\right)^{3/2}\frac{Zr}{a_0}e^{-Zr/2a_0}\cos\theta$$

$$|21\pm1\rangle \doteq \psi_{21\pm1} = \mp\frac{1}{2\sqrt{2\pi}}\left(\frac{Z}{2a_0}\right)^{3/2}\frac{Zr}{a_0}e^{-Zr/2a_0}\sin\theta\, e^{\pm i\phi}$$

etc....

Properties of $\psi_{n\ell m}$ Solutions

(2)

as these results are compiled in tables it's important for us to know how to use them rather than derive them.

(1) Normalization

$$1 = \langle n\ell m|n\ell m\rangle = \int |\psi_{n\ell m}(\theta,\phi)|^2 dV$$

which is a full 3D integral

$$\langle n\ell m|n\ell m\rangle = \int_0^\infty\int_0^{2\pi}\int_0^{\pi} |\psi_{n\ell m}(\theta,\phi)|^2 r^2\sin\theta\, d\theta\, d\phi\, dr$$

Note that $R_{n\ell}$ & Y_ℓ^m have been normalized separately.

$$\langle n\ell m|n\ell m\rangle = \underbrace{\left\{\int_0^\infty r^2|R_{n\ell}|^2 dr\right\}}_{=1}\underbrace{\left\{\int_0^{2\pi}\int_0^{\pi}|Y_{\ell m}|^2\sin\theta\, d\theta\, d\phi\right\}}_{=1}$$

Turns out that makes some calculations (3)

<u>much</u> easier.

(2) The probability density is simply the absolute square of $\psi_{n\ell m}$,

$$P(r,\theta,\phi) = |\psi_{n\ell m}(r,\theta,\phi)|^2 = |R_{n\ell}(r) Y_\ell^m(\theta,\phi)|^2$$

By multiplying by a volume element we find the probability the particle is in some volume,

$$P(r,\theta,\phi)\,dV = |\psi_{n\ell m}(r,\theta,\phi)|^2 dV$$

$$= |R_{n\ell}(r)\, Y_\ell^m(\theta,\phi)|^2 r^2 \sin\theta \, d\theta \, d\phi \, dr$$

Example: Probability of finding the electron in a sphere of radius a_0 given it is in the ground state,

$$|100\rangle \doteq \psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a_0} \qquad \left(\begin{array}{l}\text{no } z \text{ b/c} \\ \text{Hydrogen } z=1\end{array}\right)$$

④

$$\text{Prob}_{r<a_0} = \int_{\substack{\text{all } \theta,\phi \\ r<a_0}} P(r,\theta,\phi)\, dV$$

$$= \int_0^{a_0}\int_0^{2\pi}\int_0^{\pi} P(r,\theta,\phi)\, r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \int_0^{a}\int_0^{2\pi}\int_0^{\pi} \left| R_{n\ell}(r)\, Y_\ell^m(\theta,\phi)\right|^2 r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \left\{\int_0^{a} |R_{n\ell}(r)|^2 r^2 dr\right\} \left\{\underbrace{\int_0^{2\pi}\int_0^{\pi} |Y_\ell^m(\theta,\phi)|^2 \sin\theta \, d\theta \, d\phi}_{=1 \text{ just } Y_\ell^m \text{ normalization!}}\right\}$$

$$\text{Prob}_{r<a_0} = \int_0^{a} |R_{n\ell}(r)|^2 r^2 dr$$

for the ground state $n=1$, $\ell=0$,

$$= \int_0^{a} |R_{10}(r)|^2 r^2 dr = \int_0^{a} r^2 \left(\frac{4}{a_0^3}\right) e^{-2r/a_0} dr$$

Wolfram

$$\text{Prob}_{r<a_0} = 1 - 5e^{-2} \simeq 0.323$$