So we have uncovered that the radial eqn. gave rise to quantized energy and an important length scale

$$E_n = -\frac{1}{2n^2}\left(\frac{Ze^2}{4\pi\varepsilon_0}\right)\frac{\mu}{\hbar^2} \quad n = 1, 2, 3, \ldots$$

$$a \equiv \frac{4\pi\varepsilon_0\hbar^2}{\mu Z e^2}$$

For hydrogen-like atoms $m_{nucleus} \gg m_{electron}$ so $\mu \simeq m_e$ Thus we usually replace μ with m_e, the mass of the electron so,

$$\boxed{\begin{aligned} E_n &= -\frac{1}{2n^2}\left(\frac{Ze^2}{4\pi\varepsilon_0}\right)\frac{m_e}{\hbar^2} \\ a &\equiv \frac{4\pi\varepsilon_0\hbar^2}{m_e Z e^2} = \frac{a_0}{Z} \end{aligned}}$$

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where

$$\boxed{a_0 \equiv \frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2} \quad \text{is the Bohr radius}}$$

This all being said we never solved for the functions $R(\rho)$ or $R(r)$.

So let's get back to that. We had,

$$R(\rho) = \rho^{l} e^{-\gamma\rho}\left[f(\rho)\right]$$

Recalling that $\rho = r/a = Zr/a_0$

then,

$$\boxed{R_{nl} = \left(\frac{Zr}{a_0}\right)^{l} e^{-Zr/na_0}\left[f\left(\frac{Zr}{a_0}\right)\right]}$$

Recall $f(\rho) = \sum_{j=0}^{\infty} c_j \rho^j$

or $f\left(\frac{Zr}{a_0}\right) = \sum_{j=0}^{\infty} c_j \left(\frac{Zr}{a_0}\right)^j$

But j_{max} is set by n & l,

$j_{max} = n - l - 1$ so we can begin to write out solutions

Let $n=1$ and $l=0$, then $j_{max} = 0$

So that

$$R_{10}\left(\frac{Zr}{a_0}\right) = \left(\frac{Zr}{a_0}\right)^0 e^{-Zr/a_0} c_0 \left(\frac{Zr}{a_0}\right)^0$$

or

$$R_{10}(r) = c_0 e^{-Zr/a_0}$$ where c_0 is set by normalization

We can construct the remaining solutions similarly. But tables of $R_{nl}(r)$ are widely available.

E.g. normalized R_{nl}'s,

$$R_{10}(r) = 2\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

$$R_{20}(r) = 2\left(\frac{Z}{2a_0}\right)^{3/2}\left[1 - \frac{Zr}{a_0}\right] e^{-Zr/2a_0}$$

$$R_{21}(r) = \frac{1}{\sqrt{3}}\left(\frac{Z}{2a_0}\right)^{3/2} \frac{Zr}{a_0} e^{-Zr/2a_0}$$

etc...

The Associated Laguerre Polynomials

These results can be synthesized in terms of a commonly known set of functions → the associated Laguerre Polynomials

$$L_q^p(x) = \frac{d^p}{dx^p} L_q(x)$$

with

$$L_q(x) = e^x \frac{d^q}{dx^q}\left(x^q e^{-x}\right)$$

the Laguerre polynomials

So that,

$$R_{n\ell}(r) = -\left\{\left(\frac{2Z}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}\right\}^{1/2} e^{-Zr/na_0}\left(\frac{2Zr}{na_0}\right)^\ell \left(L_{n+\ell}^{2\ell+1}\left(\frac{2Zr}{na_0}\right)\right)$$

BTW I have never used this form. I just use tables to look them up.

Normalization? Each $R_{n\ell}$ is normalized

$$\int_0^\infty r^2 dr \left[R_{n\ell}(r)\right]^2 = 1$$