Angular Momentum

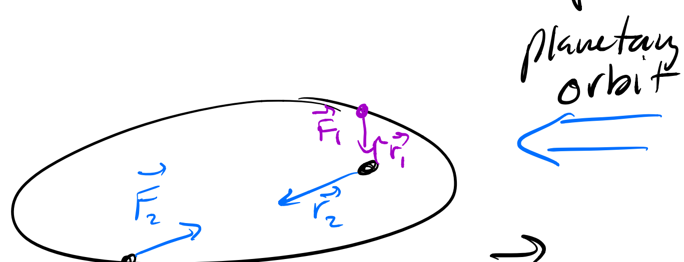
1

As our central potential problem has a lot to do with motion around the center of mass : $H_{rel}|E_{rel}\rangle = E_{rel}|E_{rel}\rangle$, the angular momentum of the system will play an important role in the physics we study.

In Classical Physics,

$$\vec{L} = \vec{r} \times \vec{p} \quad \text{and} \quad \frac{d\vec{L}}{dt} = \underset{\text{(torque)}}{\vec{\tau}} = \vec{r} \times \vec{F}$$

when the force is central $\vec{r}$ is parallel to $\vec{F}$



so that $\vec{\tau} = 0$ thus $\frac{d\vec{L}}{dt} = 0$

$\vec{L}$ is a conserved quantity!

Emmy Noether proposed when a symmetry exists there's a corresponding conservation law. ②

In our case, spherical sym $\Rightarrow$ invariant to rotations

$$\Downarrow$$

$\vec{L}$ conserved.

We can start from the classical definition to find our operators,

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$= (yp_z - zp_y)\hat{i} - (xp_z - zp_x)\hat{j} + (xp_y - yp_x)\hat{k}$$

$$= L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$$

$$L_x = yp_z - zp_y \doteq -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y = zp_x - xp_z \doteq -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$L_z = xp_y - yp_x \doteq -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Angular Momentum Operators

We can show that, 3

$$[L_x, L_y] = i\hbar L_z$$
$$[L_y, L_z] = i\hbar L_x$$
$$[L_z, L_x] = i\hbar L_y$$

Just like Spin! $[S_x, S_y] = i\hbar S_z$

We can also compute $L^2 = \vec{L}\cdot\vec{L} = L_x^2 + L_y^2 + L_z^2$ and show,

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$$

that is L^2 commutes with all components of $\vec{L}$

Like Spin! $[S^2, S_x] = 0$

So $\vec{L}$ seems to operate a lot like spin,

$$S^2 |s m_s\rangle = s(s+1)\hbar^2 |s m_s\rangle$$

$$S_z |s m_s\rangle = m_s \hbar |s m_s\rangle$$

So we can show L^2 & L_z have the same eigenvalue equations,

$$\boxed{\begin{aligned} L^2 |l m_l\rangle &= l(l+1)\hbar^2 |l m_l\rangle \\ L_z |l m_l\rangle &= m_l \hbar |l m_l\rangle \end{aligned}}$$

Big Difference: $\vec{L}$ is the angular momentum,

l is the orbital ang. mom. quantum #

$l = 0, 1, 2, \ldots$ unit steps only!

m_l is the magnetic quantum # (z-component)

$$m_l = -l, -l+1, \ldots, 0, \ldots l-1, l$$

Alot of the architecture from spin, $\vec{S}$, works for angular momentum, $\vec{L}$.

E.g. for $l=1$,

$$L^2 \doteq 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad L_z \doteq \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$L_x \doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad L_y \doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

just like spin 1 matrices!

Back to the problem at hand,

we can show,

$$L_z \doteq -i\hbar \frac{d}{d\phi}$$

15

and

$$L^2 \doteq -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right)$$

so that $H_{rel} | E_{rel} \rangle = E_{rel} | E_{rel} \rangle$ becomes,

$$-\frac{\hbar^2}{2\mu} \left(\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \underbrace{\frac{1}{\hbar^2 r^2} L^2}_{} \right) \psi + V(r)\psi = E\psi$$

the entire angular part is contained here ($\psi = R(r)\, Y(\theta, \phi)$ later)

In addition,

$$[H, L_z] = 0 \quad \& \quad [H, L^2] = 0$$

So we can find simultaneous eigenstates of H, L_z, & L^2! More on that soon.