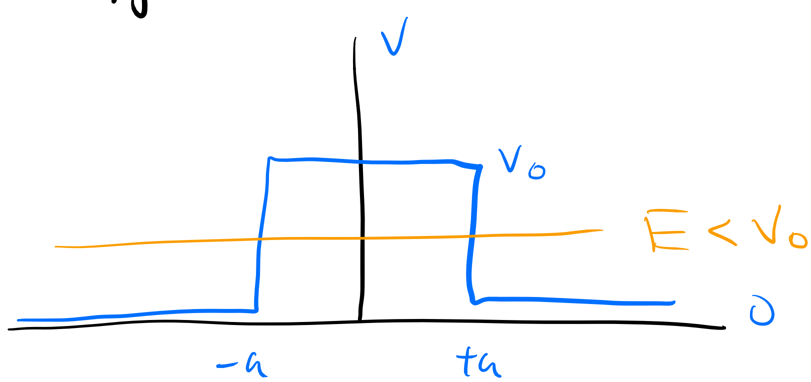
Tunneling through Barriers

One of the coolest things about QM, is there's a small, but nonzero probability for low energy particles to tunnel through a barrier.

We have built up sufficient mathematical architecture to deal with a barrier,

$$V(x) = \begin{cases} 0 & x < -a \\ +V_0 & -a < x < a \\ 0 & x > a \end{cases}$$

Here the barrier is a positive "bump" in the energy landscape.



We seek solutions with $\underline{\underline{E < V_0}}$!

We start (as usual) by writing the energy eigenvalue eqns.

$$\frac{-\hbar^2}{2m}\frac{d^2\varphi_E}{dx^2} = E\varphi_E \qquad |x| > a$$

$$\left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V_0\right)\varphi_E = E\varphi_E \qquad |x| < a$$

in this case $E > 0$ but $E < V_0$,

$|x| > a$:

$$\frac{d^2\varphi_E}{dx^2} = -\frac{2mE}{\hbar^2}\varphi_E = -k^2\varphi_E \qquad k \equiv \sqrt{\frac{2mE}{\hbar^2}} > 0$$

$|x| < a$:

$$\frac{d^2\varphi_E}{dx^2} = +\frac{2m(V_0 - E)}{\hbar^2}\varphi_E = g^2\varphi_E \qquad g \equiv \sqrt{\frac{2m(V_0-E)}{\hbar^2}} > 0$$

So we have,

$$\frac{d^2\varphi_E}{dx^2} = g^2\varphi_E \quad \text{for } |x| < a$$

$$\& \quad \frac{d^2\varphi_E}{dx^2} = -k^2\varphi_E \quad \text{for } |x| > a$$

We recast this problem as picking an energy, E, and sending in a beam amplitude A from the far left side. Then,

incident reflected

$$\varphi_E(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} \\ Ce^{gx} + De^{-gx} \\ Fe^{ikx} \end{cases}$$

can't get rid of C or D b/c finite, a

transmitted

We, again, emphasize finding

B/A & F/A b/c we want

$$T = \frac{|F|^2}{|A|^2} \quad \& \quad R = \frac{|B|^2}{|A|^2}$$

Matching conditions are still

$\varphi_E(x)$, $d\varphi_E/dx$ are continuous @ $\pm a$

$$\varphi_E(-a): \quad Ae^{-ika} + Be^{ika} = Ce^{-qa} + De^{qa}$$

$$\left.\frac{d\varphi_E}{dx}\right|_{-a}: \quad ikAe^{-ika} - ikBe^{ika} = qCe^{-qa} - qDe^{qa}$$

$$\varphi_E(+a): \quad Ce^{+qa} + De^{-qa} = Fe^{ika}$$

$$\left.d\varphi/dx\right|_{+a}: \quad qCe^{+qa} - qDe^{-qa} = ikFe^{ika}$$

Now we do the algebra to find B/A & F/A

- eliminate C & D.

$$qCe^{+qa} + qDe^{-qa} = qFe^{ika}$$

$$qCe^{+qa} - qDe^{-qa} = ikFe^{ika}$$

$$2qCe^{qa} = (q+ik)Fe^{ika} \qquad 2qDe^{-qa} = (q-ik)Fe^{ika}$$

$$\boxed{C = \left(\frac{q+ik}{2q}\right)e^{ika-qa}F \quad D = \left(\frac{q-ik}{2q}\right)e^{ika+qa}F}$$

$$ikAe^{-ika} + ikBe^{ika} = ikCe^{-qa} + ikDe^{qa}$$

$$ikAe^{-ika} - ikBe^{ika} = qCe^{-qa} - qDe^{qa}$$

$$2ikAe^{-ika} = (ik+q)Ce^{-qa} + (ik-q)De^{+qa}$$

$$2ikBe^{+ika} = (ik-q)Ce^{-qa} + (ik+q)De^{-qa}$$

$$\rightarrow 2ikAe^{-ika} = (ik+q)\left(\frac{ik+q}{2q}\right)e^{ika-qa}Fe^{-qa} - (ik-q)\frac{(ik-q)}{2q}e^{ika+qa}Fe^{qa}$$

$$2ikAe^{-ika} = \frac{F}{2q}\left((ik+q)^2 e^{ika-2qa} - (ik-q)^2 e^{ika+2qa}\right)$$

$$A = \frac{F}{4qki}\left((ik+q)^2 e^{i2ka-2qa} - (ik-q)^2 e^{i2kq+2qa}\right)$$

$$= \frac{Fe^{i2ka}}{4qki}\left((-k^2+2ikq+q^2)e^{-2qa} - (-k^2-2ikq+q^2)e^{+2qa}\right)$$

$$= \frac{Fe^{i2ka}}{4qki}\left(k^2(e^{2qa}-e^{-2qa}) + i2kq(e^{2qa}+e^{-2qa}) - q^2(e^{+2qa}-e^{-2qa})\right)$$

$$= \frac{F e^{i2ka}}{4qki}\left(2k^2 \sinh(2qa) + i4kq\cosh(2qa) - 2q^2\sinh(2qa)\right)$$

$$A = \frac{F e^{i2ka}}{4qki}\left(2(k^2-q^2)\sinh(2qa) + i4kq\cosh(2qa)\right)$$

$$\frac{F}{A} = (4qki)\left(e^{-i2ka}\right) \Big/ \left(2(k^2-q^2)\sinh(2qa) + i4kq\cosh(2qa)\right)$$

$$\frac{|F|^2}{|A|^2} = \frac{(4qki)(-4qki)\left(e^{-i2ka}\right)\left(e^{+i2ka}\right)}{\left(2(k^2-q^2)\sinh(2qa) + i4kq\cosh(2qa)\right)}$$

$$\times \frac{1}{2(k^2-q^2)\sinh(2qa) - i4kq\cosh(2qa)}$$

$$= \frac{16q^2k^2(1)}{4(k^2-q^2)^2\sinh^2(2qa) + 16k^2q^2\cosh^2(2qa)}$$

with

$$\cosh^2(x) = 1 + \sinh^2(x)$$

$$\frac{|F|^2}{|A|^2} = \frac{16q^2k^2}{4(k^2-q^2)^2\sinh^2(2qa) + 16k^2q^2 + 16k^2q^2\sinh^2(2qa)}$$

$$= \frac{16q^2k^2}{\left(4(k^2-q^2)^2 + 16k^2q^2\right)\sinh^2(2qa) + 16k^2q^2}$$

$$= \frac{16q^2k^2}{\left(4k^4 - 8k^2q^2 + 4q^2 + 16k^2q^2\right)\sinh^2(2qa) + 16k^2q^2}$$

$$= \frac{16q^2k^2}{4(k^4 + 2k^2q^2 + q^2)\sinh^2(2qa) + 16k^2q^2}$$

$$= \frac{16q^2k^2}{4(k^2+q^2)^2\sinh^2(2qa) + 16k^2q^2}$$

$$\frac{|F|^2}{|A|^2} = \frac{\cancel{16q^2k^2}}{\cancel{16k^2q^2}\left(1 + \frac{(k^2+q^2)^2\sinh^2(2qa)}{4k^2q^2}\right)}$$

$$\boxed{T = \frac{|F|^2}{|A|^2} = \frac{1}{1 + \frac{(k^2+q^2)^2 \sinh^2(2qa)}{4k^2q^2}}}$$

OOF. Now let's do B/A, if we were clever we'd just find $|B|^2/|A|^2$ instead

$$R = 1 - T = 1 - \frac{1}{1 + \frac{(k^2+q^2)^2 \sinh^2(2qa)}{4k^2q^2}}$$

We can simplify by finding a common denominator and simplifying,

$$\boxed{R = \frac{|B|^2}{|A|^2} = \frac{1}{1 + \frac{4k^2q^2}{(k^2+q^2)\sinh(2qa)}}}$$