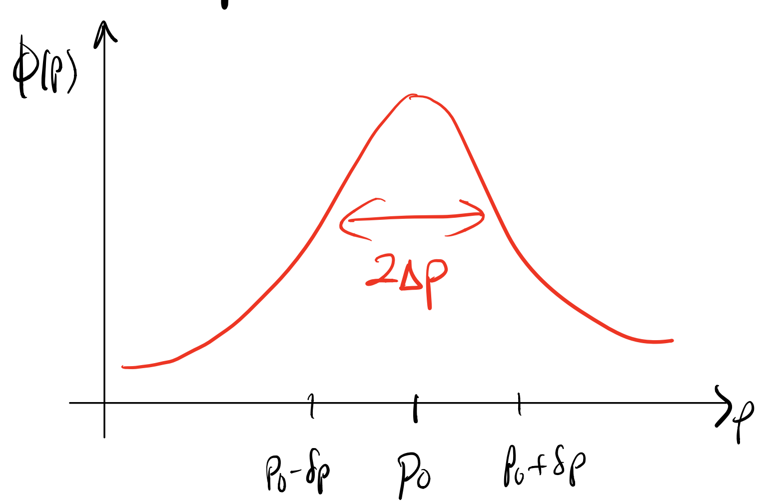
Wavepackets that we have begun to construct gives us new insight into the uncertainty principle. ①

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p)\, e^{ip(x-pt/2m)/\hbar}\, dp$$

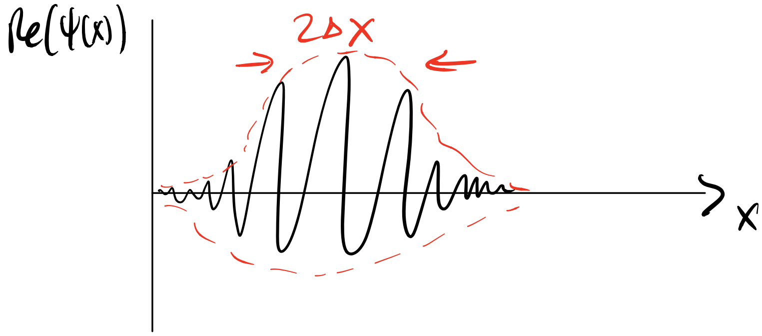
For a given distribution of momentum eigenstates, $\phi(p)$ ⇐ momentum space wavefunction, we might find some distribution that can be characterized by some extent, δp

For example a Gaussian distribution,



introduces some uncertainty in the measure of p, $\Delta p \sim \delta p$

This Δp leads to some ΔX → a wavepacket with some physical spread



The mathematical relationship that connects Δx & Δp is the Fourier (or inverse) transform of the momentum space wavefunction (or the spatial wave function).

However, the Heisenberg Uncertainty principle can help produce a <u>lower limit</u> on these uncertainties w/o much calculation.

General Heisenberg U.P.

$$\boxed{\Delta A \Delta B \geq \frac{1}{2}\left|\langle [A,B] \rangle\right|}$$

3

It can be shown that $\hat{x}$ & $\hat{p}$ do not commute,

$$[\hat{x}, \hat{p}] = i\hbar \quad \text{thus,}$$

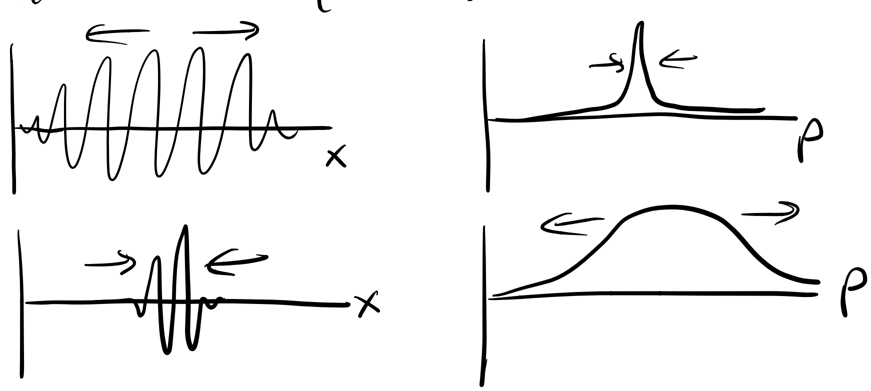
$$\Delta x \Delta p \geq \frac{1}{2} \left| \langle [\hat{x}, \hat{p}] \rangle \right| = \frac{1}{2} \left| \langle i\hbar \rangle \right|$$

$$\boxed{\Delta x \Delta p \geq \hbar/2}$$

How does this relationship manifest in our waves?

In two ways,

(1) waves with broad (narrow) spatial extents will have narrow (broad) momentum extents.



(2) in how the wavepacket evolves in time. $\Rightarrow$ it will spread out

e.g. in the derivation for the Gaussian beam, we can show,

$$\Delta x = \frac{\hbar}{2\beta}\sqrt{1+\left(\frac{2\beta^2 t}{m\hbar}\right)^2} \qquad \Delta p = \beta$$

where β is the gaussian spread in momenta

Thus,

$$\Delta x \Delta p = \frac{\hbar}{2}\sqrt{1+\left(\frac{2\beta^2 t}{m\hbar}\right)^2}$$

$t=0 \quad \Delta x \Delta p = \hbar/2 \qquad t>0 \quad \Delta x \Delta p > \hbar/2$

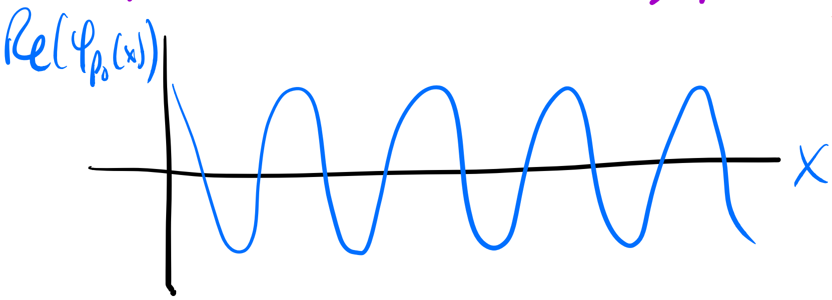
Example 1: A single momentum state

Let's consider a single momentum eigenstate to see how $\Delta x \Delta p \geq \hbar/2$ plays out.

A single momentum eigenstate is a pure sinusoidal state in position space.

$$\langle x|p_0\rangle = \varphi_{p_0}(x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{ip_0x/\hbar}$$

Note: here we have spatial representation of a pure sinusoidal function given a particular momentum, p_0



Infinite extent

Let's use the Fourier transform to find the representation in momentum space.

$$\phi_{p_0}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \varphi_{p_0}(x)\, e^{-ipx/\hbar}\, dx$$

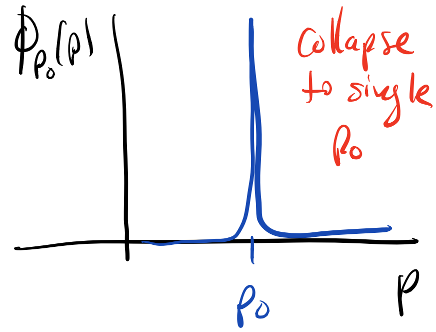
$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{+ip_0x/\hbar}\, e^{-ipx/\hbar}\, dx$$

$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{-i(p-p_0)x/\hbar}\, dx$$

6

This integral form is precisely the one for Dirac normalization,

$$\langle p | p_0 \rangle = \delta(p - p_0)$$



So,

$$\phi_{p_0}(p) = \delta(p - p_0)$$

Infinite spatial extent $\Longleftrightarrow$ unique p

Example 2: Particle state

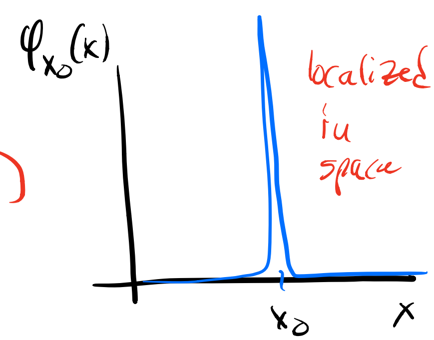
A particle state would have a perfectly identifiable location, say x_0. thus

$\langle x | x_0 \rangle = \varphi_{x_0}(x) = \delta(x - x_0)$ is the position representation of this particle's state.

Note: $\hat{x}|x_0\rangle = x_0 |x_0\rangle$

$\hat{x}\,\delta(x - x_0) = x_0\,\delta(x - x_0)$

are the eigenvalue eqns.



So the Fourier transform of $\varphi_{x_0}(x)$ (7)

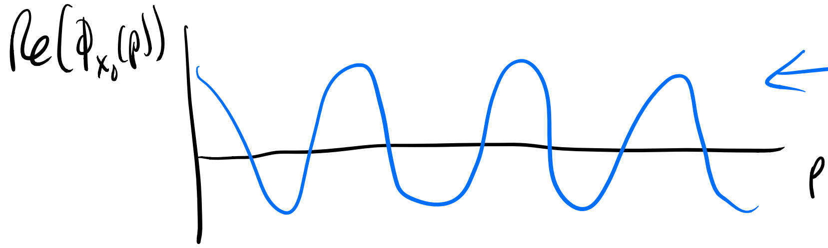
will give us the momentum representation,

$$\phi_{x_0}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \varphi_{x_0}(x)\, e^{-ipx/\hbar}\, dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \delta(x-x_0)\, e^{-ipx/\hbar}\, dx$$

$$\phi_{x_0}(p) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{-ipx_0/\hbar}$$

recall that p is the variable in $\phi_{x_0}(p)$



Infinite extent!

highly localized spatial extent $\Longleftrightarrow$ infinite momentum extent.