In order to more easily describe wave packets it will be useful to work with the momentum distribution. In addition, we will find there's a well established way for getting from the momentum representation to the position representation for the free particle. So we will start from the wave vector eigenstates, (1)

$$\boxed{\psi_k(x) = A e^{ikx} \qquad -\infty < k < \infty}$$

Let's operate on ψ_k with $\hat{p}$,

$$\hat{p}\,\psi_k(x) = \left(-i\hbar\frac{d}{dx}\right)\psi_k(x)$$

$$= \left(-i\hbar\frac{d}{dx}\right)\left(Ae^{ikx}\right) = -i\hbar A\frac{d}{dx}\left(e^{ikx}\right)$$

$$= (-i\hbar)(ik)Ae^{ikx} = \hbar k A e^{ikx}$$

$$\boxed{\hat{p}\,\psi_k(x) = \hbar k\,\psi_k(x)}$$

Because operating on $\psi_k(x)$ with $\hat{p}$ gives

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us a constant times $\varphi_k(x)$, then we know that the wave vector eigenstates are also momentum eigenstates!

$$\hat{p}\,\varphi_k(x) = \hbar k\,\varphi_k(x)$$

$$\hat{p}\,\varphi_p(x) = p\,\varphi_p(x) \qquad \left[\text{or think, } \hat{p}|p\rangle = p|p\rangle\right]$$

So, $\boxed{p = \hbar k}$

and the momentum eigenstates are,

$$\boxed{\varphi_p(x) = A e^{ipx/\hbar} \quad \text{where } -\infty < p < \infty}$$

In modern physics you learned about wave-particle duality likely first by learning the deBroglie relationship $\boxed{\lambda = h/p}$

this inference from late 19th century & early 20th century physics falls out of the free particle results,

$k = 2\pi/\lambda \Leftarrow$ from wave mechanics and $p = \hbar k = \frac{h}{2\pi}k \Leftarrow$ from free particle

Gives $\Rightarrow$ $$p = \frac{h}{2\pi}\left(\frac{2\pi}{\lambda}\right) = \frac{h}{\lambda} \quad \text{or} \quad \underline{\lambda = h/p}$$ 3

Energy Eigenstates and Time Evolution

B/c there's no potential ($V(x)$) in the Hamiltonian of the free particle,

$$\hat{H} = \hat{p}^2/2m$$

The momentum eigenstates, $\varphi_p(x)$, are also energy eigenstates! (with eigenvalue $E_p = p^2/2m$)

So time evolution is quite straightforward,

$$\psi_p(x,t) = \varphi_p(x)\, e^{-iE_p t/\hbar}$$

$$= A e^{ipx/\hbar} e^{-i\frac{p^2}{2m\hbar}t}$$

$$\boxed{\psi_p(x,t) = A e^{i\frac{p}{\hbar}\left(x - \frac{p}{2m}t\right)}}$$

Notice this has the form $f(x - vt)$ where $|v| = p/2m$ half the classical speed.
B/c this is the "phase velocity".

Individual Momentum Eigenstates are not QM normalizable

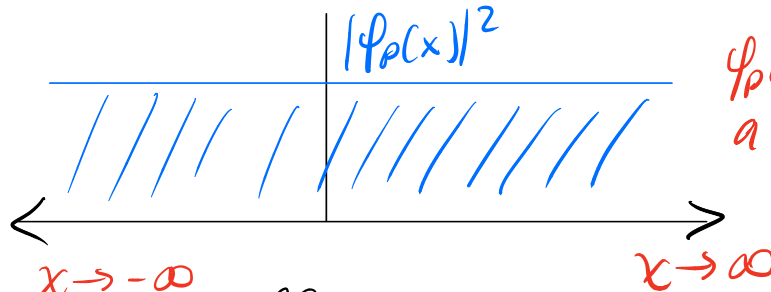
Let's go back to looking at a given momentum eigenstate,

$$\varphi_p(x) = Ae^{ipx}$$

Let's compute the probability density,

$$P(x) = |\varphi_p(x)|^2 = \varphi_p^*(x)\,\varphi_p(x)$$

$$= |A|^2 e^{-ipx} e^{+ipx} = |A|^2 \quad \underline{\text{CRAP!}}$$



$\varphi_p(x)$ gives a uniform probability!

$$\int_{-\infty}^{\infty} P(x)\,dx \Rightarrow \infty !!$$

Every Basis we have used so far has had 3 properties,

$$\langle a_i | a_j \rangle = \delta_{ij} \quad \text{orthogonal (1) \& normal (2)}$$

$$\sum_i |a_i\rangle\langle a_i| = 1 \quad \text{complete (3)}$$

What the heck do we do with $\psi_p(x)$ then?

- We typically will work with a distributions of momentum eigenstates $\Rightarrow$ this turns out to solve our mathematical problem & it is also experimentally valid as there's usually some distribution of momenta.
- To do this we will need to adapt our properties to our new continuous basis.

Orthonormality

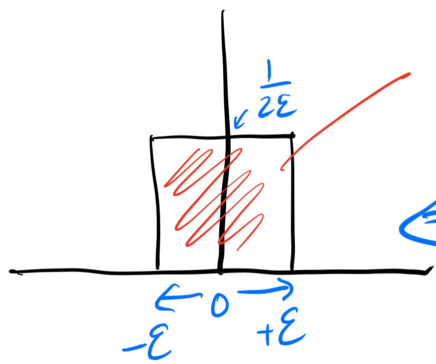
To adapt $\langle a_i | a_j \rangle = \delta_{ij}$ to a continuous basis, we introduce the Dirac Delta function.

- the Dirac-δ is an infinitely thin, infinitely tall function located at a given location. (in the case of $\delta(x-x_0)$ it's located at $x=x_0$)

The critical property of the Dirac-δ is that it's integral is one

$$\boxed{\int_{-\infty}^{\infty} \delta(x-x_0)\,dx = 1}$$

Conceptually, the Dirac-δ is the limit of shrinking width, growing height uniform distribution,



$$A = \left(\frac{1}{2\varepsilon}\right)\left(+\varepsilon-(-\varepsilon)\right) = \frac{1}{2\varepsilon}(2\varepsilon) = 1$$

In the limit $\varepsilon \to 0$, we have a δ function at $x=0$.

Returning to Momentum Eigenstates, our new orthonormality condition is,

$$\langle p'' | p' \rangle = \delta(p''-p')$$

or

$$\int_{-\infty}^{\infty} \varphi^*_{p''}(x)\, \varphi_{p'}(x)\, dx = \delta(p''-p')$$

* with $\varphi_{p'}(x) = Ae^{ip'x/\hbar}$ and $\varphi_{p''}(x) = Ae^{ip''x/\hbar}$

↳ We find we can normalize $\varphi_p(x)$ with $A = \frac{1}{\sqrt{2\pi\hbar}}$

* this proof relies on a doing a Fourier transform (actually an inverse transform), which you will walk through in your homework.

A "Dirac" Normalized Momentum Eigenstate

$$\boxed{\varphi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{ipx/\hbar}}$$

Completeness

We have understood completeness as being able to write any general state vector as a linear combination of basis states,

$$\sum_i |a_i\rangle\langle a_i| = 1$$

$$|\psi\rangle = \sum_i |a_i\rangle\langle a_i|\psi\rangle = \sum_i a_i |a_i\rangle$$

coefficients for each basis vector

In a continuous basis (like the momentum eigenstates of the free particle), we have to add up over all possible momentum states .... so $\sum_i \rightarrow \int dp$

$$\boxed{1 = \int_{-\infty}^{+\infty} |p\rangle\langle p|\, dp}$$

Completeness in F.P. momentum eigenstates

We can use the completeness relationship to express any general state in the momentum basis,

$$\psi(x) = \langle x | \psi \rangle = \langle x | 1 | \psi \rangle$$

$$\psi(x) = \langle x | \left\{ \int_{-\infty}^{\infty} |p\rangle\langle p| \, dp \right\} | \psi \rangle$$

$$\psi(x) = \int_{-\infty}^{\infty} \underbrace{\langle x | p \rangle}_{\text{projection of the momentum eigenstate onto position basis, } \varphi_p(x)} \underbrace{\langle p | \psi \rangle}_{\text{projection of } \psi \text{ onto the momentum basis}} \, dp$$

What we have already seen.

$\phi(p)$ the momentum space wavefunction.

$$\boxed{\psi(x) = \int_{-\infty}^{\infty} \varphi_p(x) \, \phi(p) \, dp \qquad \psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p) \, e^{ipx/\hbar} \, dp}$$

a general state written in the momentum basis

To go any further, we need $\langle p | \psi \rangle = \phi(p)$.

Fourier Transforms

Now we see again why our initial choice of e^{ikx} paid off. $\psi(x)$ is just the Fourier Transform of $\phi(p)$

$$\boxed{\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \phi(p)\, e^{ipx/\hbar}\, dp}$$

That means $\phi(p)$ is obtained by the inverse Fourier transform of $\psi(x)$!

$$\boxed{\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x)\, e^{-ipx/\hbar}\, dx}$$