Until now, we have worked with systems (1)
that give rise to quantized energy
states. This has been the result
of either:

(1) The system having intrinisic spin,
which is a fundamentally QM
property (Chs. 1–3 of McIntyre)

OR

(2) The system having a low enough
energy such that it exhibits
boundstates (Ch. 5. of McIntyre)

We now turn to a system that has
neither of these properties, but we
still describe it using our QM tools.

The Free Particle is subject to no
potential and thus can take on
any energy ⟹ it is no longer Quantized.

The Free Particle

We will begin by describing the free particle w/ our QM tools.

We use the same eigenvalue equation,

$$\hat{H}|E\rangle = E|E\rangle$$

With $\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{x})$, 1D Hamiltonian

$$\hat{H}\varphi_E(x) = E\varphi_E(x)$$ 1D eigenvalue problem

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\varphi_E(x) = E\varphi_E(x)$$

For the Free Particle $V(x) = 0$ everywhere!

so

$$\frac{d^2}{dx^2}\varphi_E(x) = \underbrace{-\frac{2mE}{\hbar^2}}_{}\varphi_E(x)$$

All positive quantities

so $k^2 \equiv 2mE/\hbar^2$

$$\frac{d^2\varphi_E(x)}{dx^2} = -k^2\varphi_E(x)$$

3

General solution:

$$\varphi(x) = Ae^{ikx} + Be^{-ikx} \qquad \boxed{k \geq 0}$$

- In this situation, we are effectively done. We have no boundary conditions to constrain A, B, & k. All we have is the normalization condition.
- However, as there are no constraints of k, say due to potential wells, it appears that any value of $k \geq 0$ is ok. And thus any energy, $E \geq 0$, is allowed.
- We could have chose $A\cos(kx) + B\sin(kx)$ or $D\sin(kx+\delta)$, etc. But we will see the mathematical advantage of using e^{ikx} as we explore the free particle.

Energy Eigenstates & Time Evolution

One of the more curious aspects of the free particle is that E is continuous, so that any E appearing in:

$$\hat{H}\varphi_E(x) = E\varphi_E(x)$$

is an eigenvalue of $\hat{H}$, so long as

$$\varphi_E(x) = Ae^{ikx} + Be^{-ikx} \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

Corresponds to the same E.

This means the general solution represents the energy eigenstates,

$$\boxed{\varphi_E(x) = Ae^{ikx} + Be^{-ikx}}$$

So time Evolution is simply multiplying by $e^{-iEt/\hbar}$

$$\boxed{\begin{aligned} \psi_E(x,t) &= \varphi_E(x)\, e^{-iEt/\hbar} \\ \psi_E(x,t) &= \left(Ae^{ikx} + Be^{-ikx}\right)e^{-iEt/\hbar} \end{aligned}}$$

Now, we begin to see why our choice of e^{ikx} is beneficial $\Rightarrow$ our solution leads to a classical wave form!

With $E = \hbar\omega$,

$$\psi_E(x,t) = (Ae^{ikx} + Be^{-ikx})\,e^{-i\omega t}$$

$$\psi_E(x,t) = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}$$

$$\boxed{\psi_E(x,t) = Ae^{ik\left(x-\frac{\omega t}{k}\right)} + Be^{-ik\left(x+\frac{\omega t}{k}\right)}}$$

This wave function has the form, $f(x \pm vt)$ where $|v| = \omega/k$ ← phase velocity → more later.... and is the sum of a right $(+x)$ propagating wave (A) and a left $(-x)$ propagating wave (B).

We will often allow k to run from $-\infty$ to $+\infty$, in that case,

$$\boxed{\varphi_k(x) = Ae^{ikx} \qquad -\infty < k < \infty}$$

are the wave vector eigenstates