Operators, Eigenvalues, & Eigenvectors

As we have begun to build up our understanding of kets and linear algebra representation of QM, we begin to extend it by understanding that much of QM is solving the "Eigenvalue Problem".

- Two critical postulates result in the Eigenvalue Problem framing.

Postulate 2: a physical observable is represented by an operator acting on a ket

Postulate 3: the only possible measurement of an observable is an eigenvalue of the operator.

Let's start with the eigenvalue equations for our spin 1/2 system:

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$$S_z |+\rangle = +\frac{\hbar}{2} |+\rangle$$

operator ket eigenvalue ket

In words, acting on the $|+\rangle$ ket with the S_z operator results in the eigenvalue $+\hbar/2$ times the $|+\rangle$ ket.

Or.... if we measure S_z for the $|+\rangle$ ket we get $+\hbar/2$.

Similarly for the $|-\rangle$ ket,

$$S_z |-\rangle = -\frac{\hbar}{2} |-\rangle$$

Matrix Representations

As we saw earlier we can represent kets as column vectors,

$$|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now, because the S_z operator for spin $1/2$ produces <u>two</u>

eigenvalues it must be represented (3)
by a square 2x2 matrix.

$$S_z \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

In addition because it produces real eigenvalues it must also be Hermitian. (More on that later)

With $S_z \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we can determine the elements by working through the eigenvalue problem

(1) $S_z|+\rangle \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(2) $S_z|-\rangle \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Eqn 1 gives, $a = +\frac{\hbar}{2}$ $c = 0$ 4

Eqn 2 gives, $b = 0$ $d = -\frac{\hbar}{2}$

So that, $$\boxed{S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

A couple things to note.

1. S_z is represented by a diagonal Matrix.
   This is because an operator (S_z) is always diagonal in its own basis (also S_z).
   This allows you to read off the eigen values.
2. S_z is Hermitian as its eigenvalues are real.
   Hermitian means that the Complex conjugate transpose of S_z

is the same as the original matrix. 5

$$S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \xrightarrow{\text{complex conj.}} \overline{S_z} \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\downarrow \text{transpose}$$

$$\overline{S_z^T} \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

equal

b/c S_z is diagonal and the elements are real, it automatically satisfies this.

Spin Operators

We won't derive the other operators, but we can show that $S_x = \overline{S_x^T}$ & $S_y = \overline{S_y^T}$.

$$S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow{\text{conj.}} \overline{S_x} \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow{\text{trans.}} \overline{S_x^T} \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \xrightarrow{\text{conj.}} \overline{S_y} \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \xrightarrow{\text{trans.}} \overline{S_y^T} \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The complex conjugate transpose is called the <u>adjoint</u>.

$S_z^\dagger$ (S_z "dagger") $\quad S_z = S_z^\dagger$

So,

$$\boxed{S_x \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_z \doteq \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

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Operators more generally

- operators have the same dimensions as the basis of the system under investigation. ("Hilbert Space")
- For spin 1/2 → 2 eigenvalues → 2x2
  spin 1 → 3 eigenvalues → 3x3
  spin 3/2 → 4 eigenvalues → 4x4
  etc.

What characterizes the operator in its matrix representation are the values that it carries → it matrix elements

a general matrix element is given by

$$\langle \text{bra} | \text{OPERATOR} | \text{ket} \rangle$$

For a spin $1/2$ system with the usual basis $|+\rangle, |-\rangle$ an operator would be represented as,

$$A \doteq \begin{pmatrix} \langle +|A|+\rangle & \langle +|A|-\rangle \\ \langle -|A|+\rangle & \langle -|A|-\rangle \end{pmatrix}$$

For a spin 1 system, we might have a basis like $|+\rangle, |0\rangle, |-\rangle$. So

eigenvalues $\rightarrow$ $+\hbar$ $\quad 0\hbar \quad -\hbar$

the operator A is represented,

$$A \doteq \begin{pmatrix} \langle +|A|+\rangle & \langle +|A|0\rangle & \langle +|A|0\rangle \\ \langle 0|A|+\rangle & \langle 0|A|0\rangle & \langle 0|A|-\rangle \\ \langle -|A|+\rangle & \langle -|A|0\rangle & \langle -|A|-\rangle \end{pmatrix}$$

All of this work to develop these representations leads us to an important idea in QM.

Big Idea: Diagonalization of Operators

QM operators that represent physical observables have real eigenvalues. The process of Diagonalization lets you find the eigenvalues and eigenvectors of the operator. This is very important for determining energy spectra (energy eigenvalues) and the associated states (energy eigenstates)

Example: Eigenvalues & Eigenstates of S_y

What are the eigenvalues & eigenstates of S_y?

$$S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Solution: This is a two step process (λ, then $|\lambda\rangle$)

First, we setup the eigenvalue eqn. where λ are the unknown eigenvalues and $|\lambda\rangle$ are the unknown eigenstates.

$$S_y|\lambda\rangle = \lambda|\lambda\rangle$$

$$(S_y - \lambda)|\lambda\rangle = 0$$

Solutions to this equation only exist if

$$\det|S_y - \lambda I| = 0 \quad \text{where } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So we solve this,

$$\det\left|\begin{pmatrix} 0 & -i\hbar/2 \\ i\hbar/2 & 0 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0$$

$$\det\begin{vmatrix} -\lambda & -i\hbar/2 \\ i\hbar/2 & -\lambda \end{vmatrix} = 0$$

So,

$$\lambda^2 - \frac{\hbar^2}{4} = 0 \qquad \boxed{\lambda = \pm\frac{\hbar}{2} \quad \text{(as expected)}}$$

So we got two eigenvalues as we should have gotten and they are distinct ("not degenerate").

Each eigenvalue gives rise to an eigenstate, So we now use each λ, we can find (10) the associated eigenstates, $|\lambda\rangle$.

With $\lambda = +\frac{\hbar}{2}$ $\quad |\lambda\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ (unknown a & b)

$$S_y |\lambda\rangle = +\frac{\hbar}{2} |\lambda\rangle$$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} -ib \\ ia \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{so,} \quad b = ia$$

in both cases

This isn't enough information to determine a & b. But the ket $|\lambda\rangle$ needs to be normalized.

$$\langle\lambda|\lambda\rangle = |a|^2 + |b|^2 = 1$$

$$= |a|^2 + |ia|^2 = 2a^2 = 1 \quad a = \frac{1}{\sqrt{2}}$$

So that,

$$b = \frac{i}{\sqrt{2}}$$

Thus the eigenstate is $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

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B/c this is for S_y & $\lambda = +\hbar/2$ we label it with the ket $|+\rangle_y$.

$$\boxed{|+\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}}$$

We can perform similar calculations for $\lambda = -\hbar/2$ and for S_x. We would end up with,

$$\boxed{\begin{array}{ll} |+\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} & |-\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} \\ |+\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} & |-\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix} \end{array}}$$

Projection Operators

- Now that we have built up this structure we can understand how some of the more confusing QM experiments produce the measurements we have seen. 12

- The projection operators we will develop provide the mathematical foundation that explain these experiments.

Let's look at how we construct these operators for a spin 1/2 system.

Starting with a general state in the S_z basis,

$$|\psi\rangle = a|+\rangle + b|-\rangle$$

coeffs.

$$= (\langle +|\psi\rangle)|+\rangle + (\langle -|\psi\rangle)|-\rangle$$

We can move the kets around,

$$|\psi\rangle = |+\rangle\langle +|\psi\rangle + |-\rangle\langle -|\psi\rangle$$

$$= \underbrace{\left(|+\rangle\langle +| + |-\rangle\langle -|\right)}_{\text{must be equal to } 1!}|\psi\rangle$$

$$\boxed{|+\rangle\langle+| + |-\rangle\langle-| = 1}$$ completeness relationship

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the matrix representation of this is formed by computing the "outer product",

$$|+\rangle\langle+| + |-\rangle\langle-| \doteq \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$|+\rangle\langle+| + |-\rangle\langle-| = 1$$ identity matrix.

The individual terms are the "projection operators" for $|+\rangle$ & $|-\rangle$

$$\boxed{\begin{aligned} P_+ &= |+\rangle\langle+| \doteq \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \\ P_- &= |-\rangle\langle-| \doteq \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\end{aligned}}$$

the tail P means projection not probability

Why is this useful? The projection operator is like a "dot product". It returns a ket aligned with the projection eigenstate and gives the amplitude and phase for that state to be measured in the associated eigenstate.

$$P_+|\psi\rangle = |+\rangle\langle +|\psi\rangle = \underbrace{(\langle +|\psi\rangle)}_{\text{Amp \& phase, } a}|+\rangle$$

$$P_-|\psi\rangle = |-\rangle\langle -|\psi\rangle = \underbrace{(\langle -|\psi\rangle)}_{\text{Amp \& phase, } b}|-\rangle$$

Relationship to Measurement

As we have seen, a measurement results in finding a system in a particular state with a particular probability.

We can determine that ket from projection operators. (postulate 5)

Probability$_+ = P_+ = |\langle + | \psi \rangle|^2$ no tail

$$= \langle + | \psi \rangle^* \langle + | \psi \rangle = \langle \psi | + \rangle \langle + | \psi \rangle$$

$$= \langle \psi | P_+ | \psi \rangle$$

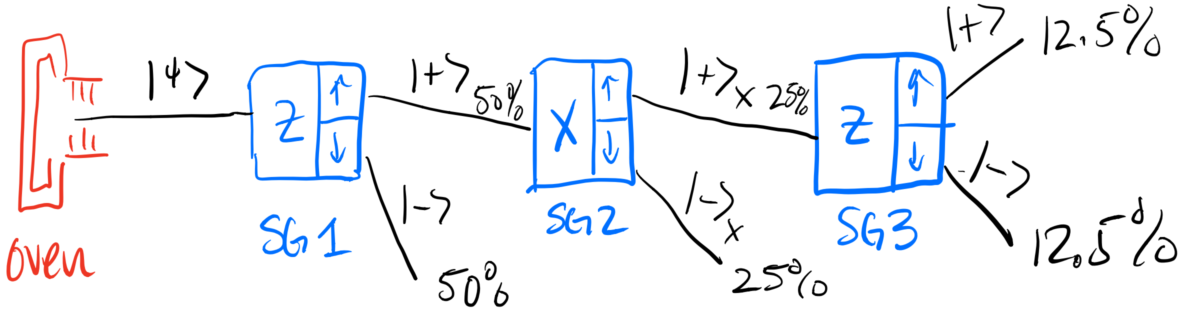
Thus, the new state is given by,

$$|\psi'\rangle = \frac{P_+ |\psi\rangle}{\sqrt{\langle \psi | P_+ | \psi \rangle}} = |+\rangle$$

or in general for P_n (some projector.)

$$\boxed{|\psi'\rangle = \frac{P_n |\psi\rangle}{\sqrt{\langle \psi | P_n | \psi \rangle}}}$$

In 2.2.4, McIntyre uses this framework to show how this experiment works.



Expectation Values

- As QM is entirely probablistic, it makes sense to understand what we might expect (on average) from a given operator for a given system.

This approach is based on probability theory where the mean value is determined by a weighted average. For S_z,

$$\langle S_z \rangle = +\frac{\hbar}{2} P_+ + \left(-\frac{\hbar}{2}\right) P_-$$

probability

as $P_+ = |\langle + | \psi \rangle|^2$ and $P_- = |\langle - | \psi \rangle|^2$

$$\langle S_z \rangle = +\frac{\hbar}{2} |\langle + | \psi \rangle|^2 + \left(-\frac{\hbar}{2}\right) |\langle - | \psi \rangle|^2$$

$$= +\frac{\hbar}{2} \langle \psi | + \rangle \langle + | \psi \rangle + \left(-\frac{\hbar}{2}\right) \langle \psi | - \rangle \langle - | \psi \rangle$$

$$= \langle \psi | \left[+\frac{\hbar}{2} | + \rangle \langle + | \psi \rangle + \left(-\frac{\hbar}{2}\right) | - \rangle \langle - | \psi \rangle \right]$$

$$= \langle \psi | \left[S_z | + \rangle \langle + | \psi \rangle + S_z | - \rangle \langle - | \psi \rangle \right]$$

$$= \langle \psi | S_z \underbrace{\left[|+\rangle\langle +| + |-\rangle\langle -| \right]}_{1} |\psi\rangle$$ 17

$$\boxed{\langle S_z \rangle = \langle \psi | S_z | \psi \rangle}$$

expectation value using operators & state vectors

In general, we can find the expectation value of any operator using either approach,

$$\boxed{\langle A \rangle = \langle \psi | A | \psi \rangle = \sum_n P_n a_n}$$

Example: Computing $\langle S_z^2 \rangle$ for $|+\rangle$ & $|-\rangle$

Let's compute the expectation value of the operator S_z^2 for both $|+\rangle$ & $|-\rangle$.

for $|+\rangle$,

$$\langle S_z^2 \rangle = \langle + | S_z^2 | + \rangle = \langle + | S_z S_z | + \rangle$$

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$$\langle S_z^2 \rangle = \langle + | S_z \tfrac{\hbar}{2} | + \rangle = \tfrac{\hbar}{2} \langle + | S_z | + \rangle$$

$$\langle S_z^2 \rangle = \frac{\hbar^2}{4} \langle + | + \rangle = \frac{\hbar^2}{4}$$

For $|-\rangle$,

$$\langle S_z^2 \rangle = \langle - | S_z^2 | - \rangle = \langle - | S_z S_z | - \rangle$$

$$\langle S_z^2 \rangle = \langle - | S_z (-\tfrac{\hbar}{2}) | - \rangle = -\tfrac{\hbar}{2} \langle - | S_z | - \rangle$$

$$= \frac{\hbar^2}{4} \langle - | - \rangle = \frac{\hbar^2}{4}$$

Commutation

Finally, lets look into why we are unable to measure S_x, S_y, and S_z at the same time.

Commuting is something we take for granted in algebra. You would find it bonkers if someone told you that

$$5x \neq x5$$

But my oldest kid still doesn't belive these two equations are the same.

Her skepticism is what you need to bring to QM b/c here we have non commutative algebra. And this commutation is central to whether two observables can be measured together.

The commutator is given by,

$$[A,B] = AB - BA$$

In most of your experience so far,

$$[A,B] = 0 \qquad AB - BA = 0$$

and thus $AB = BA$

this is how regular algebra works.

When two operators commute they share eigenstates (i.e. one measurement is a proxy for the other), so you can measure both observables together.

Assume $AB = BA$,

Let $A|a\rangle = a|a\rangle$ eigenvalue & eigenstate.

then,

$$BA|a\rangle = Ba|a\rangle$$

and

$$AB|a\rangle = aB|a\rangle$$

so,

$$A\left(B|a\rangle\right) = a\left(B|a\rangle\right)$$

the state $B|a\rangle$ is an eigenstate of A we can measure A & B "at the same time".

Non commuting Observables

This is far more common than you think in QM. Take S_x & S_z for example,

$$[S_z, S_x] = S_z S_x - S_x S_z$$

$$= \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \left(\frac{\hbar}{2}\right)^2 \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

$$= \left(\frac{\hbar}{2}\right)^2 \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] = \left(\frac{\hbar}{2}\right)^2 \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

$$S_y \doteq \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

so that

$$[S_x, S_z] = i\hbar S_y$$

In fact,

$$\boxed{\begin{aligned} [S_x, S_y] &= i\hbar S_z \\ [S_y, S_z] &= i\hbar S_x \\ [S_z, S_x] &= i\hbar S_y \end{aligned}}$$

these commutation relations are important b/c they inform the precision of potential measurement through the uncertainty principle

For example,

$$\underbrace{\Delta S_z}\,\underbrace{\Delta S_x} \geq \frac{1}{2}\left|\langle [S_z, S_x] \rangle\right|$$

uncertainty in measurements

$$\geq \frac{1}{2}\left|\langle i\hbar S_y \rangle\right|$$

$$\geq \frac{\hbar}{2}\left|\langle S_y \rangle\right|$$

$$\boxed{\Delta S_z \Delta S_x \geq \left(\frac{\hbar}{2}\right)^2}$$

Or in general

$$\boxed{\Delta A \Delta B \geq \frac{1}{2}\left|\langle [A, B] \rangle\right|}$$