Quantum Mechanics

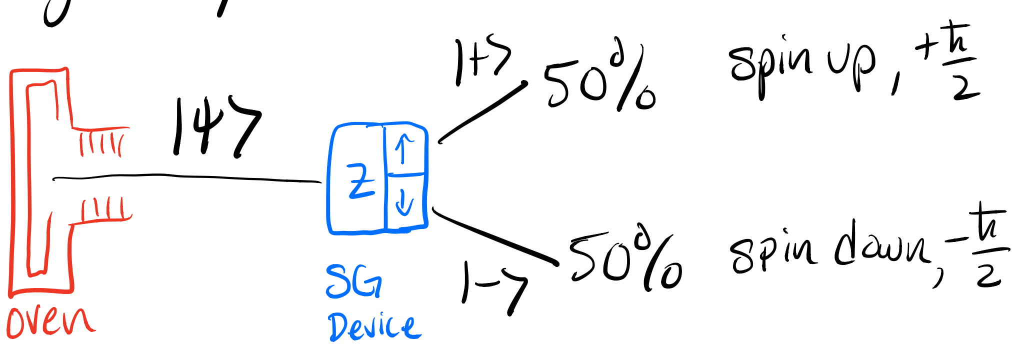
- QM is how we describe the properties, interactions, and measurements of the smallest things in the universe.
- The QM framework is fundamentally built on the idea the we can only describe systems probabilitistically.
- It brings together sophisticated mathematical ideas into its formalism, concepts from linear algebra, differential equations, complex analysis, & probability.

In this first set of lectures, we will focus on reminding ourselves of the formalism and general approach to solving QM problems by focusing on 2 state systems.

Stern Gerlach Experiments

- SG Experiments helped us understand the nature of spin angular momentum.
- Focusing on "spin ½" particles, we find that measurements of the "z component" of the spin results in precisely two values $\pm \hbar/2$ where $\hbar = 1.0546 \cdot 10^{-34}\, J \cdot s$

this measurements are summarized in the following diagram.



Here an oven produces atoms in a beam described by the general state vector, $|\psi\rangle$.

$|\psi\rangle$ is a ket. It describes all the possible information you know about a given QM system. (Postulate 1)

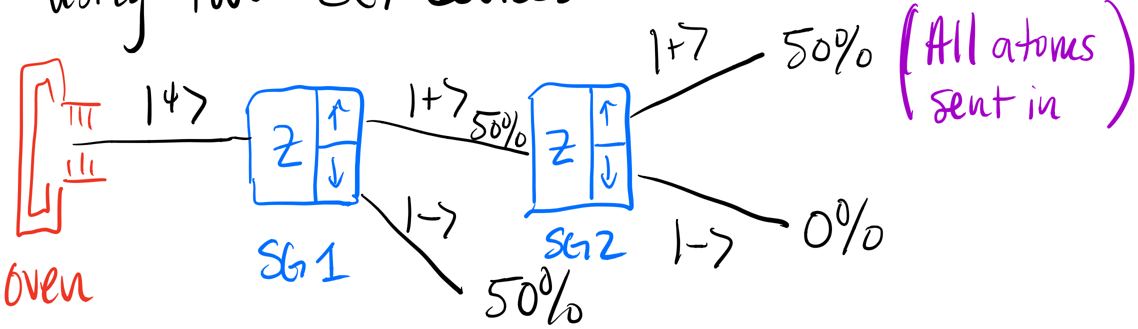
As this beam goes through the SG device (a magnetic field), we find that 2 beams emerge. These beams are characterized by having one of two possible spin measures. (S_z)

① $+\frac{\hbar}{2}$, spin up, $|+\rangle$ ⇐ ket indicates spin up (z-basis)

② $-\frac{\hbar}{2}$, spin down, $|-\rangle$ ⇐ ket indicates spin down (z-basis)

Notice that for a general ket, $|\psi\rangle$, the probability of observing either $|+\rangle$ or $|-\rangle$ is 0.5 for this experiment.

Additional evidence for this is shown by using two SG devices.



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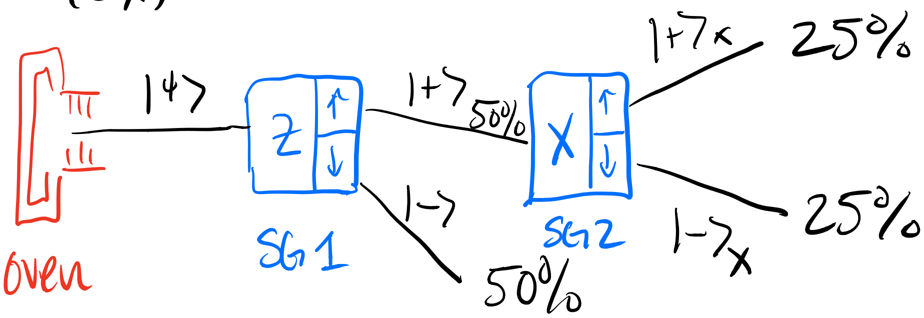
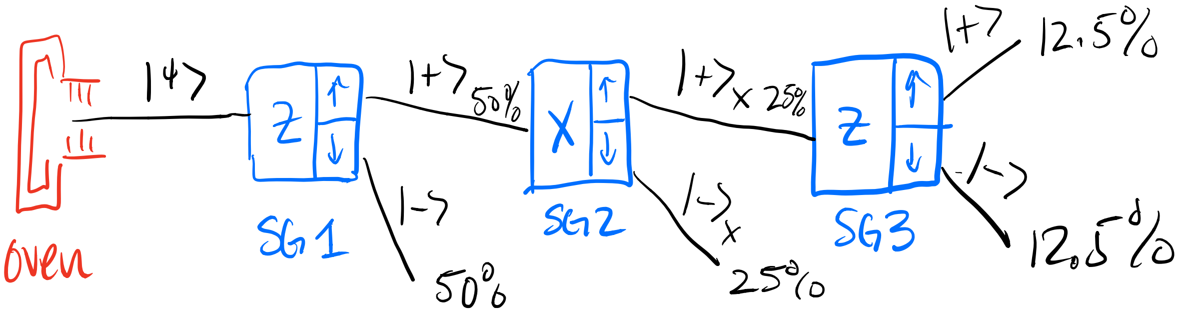
- This experiment indicates the beam sent to SG2 is a pure $|+\rangle$ beam. All atoms have a $+\frac{\hbar}{2}$ spin projection in z

Things become a little odd if we instead swap SG2 for an SG device that measures the x-component of the spin angular momentum. (S_x)



Evidentally a pure $|+\rangle$ beam with $S_z = +\hbar/2$ will still produce beams with $S_x = \pm\hbar/2$ in equal measure.

This is our first suggestion that something about QM is different. But the next experiment where we add another SG device to the mix really is striking.



So we sent a pure $|+\rangle$ beam to SG2 and then a pure $|+\rangle_x$ beam to SG3. But somehow we got $|+\rangle$ and $|-\rangle$ beams from SG3 with equal probability!

Evidentally measuring S_x makes the beam "forget" about S_z!

* This is meta physics, BTW. Really what happens is by making a quantum measurement we disturb the system resulting in a pure $|+\rangle_x$ state, which is a superposition of $|+\rangle$ and $|-\rangle$.

As we learned S_x & S_z are incompatible observables. We cannot definitively know the value of S_x & S_z for any quantum system. (6)

Quantum State Vectors

For the moment we limit our discussion to Spin 1/2 systems.

- A general quantum state vector is the linear combination of the two basis kets.
- We typically choose S_z to be our basis.

complex #'s

$$|\psi\rangle = a|+\rangle_z + b|-\rangle_z = a|+\rangle + b|-\rangle$$

we drop "z" b/c we understand that we work in the S_z basis

It is helpful if the basis we use to describe a general QM state is orthogonal, normal, & complete. There are nice things we can do with such bases.

To be able to investigate such properties, we introduce the "bra" as in "bra-ket".

For the ket $|\psi\rangle = a|+\rangle + b|-\rangle$, the corresponding bra is the complex conjugate transpose,

$$\langle\psi| = a^*\langle+| + b^*\langle-|$$

complex conjugates of a & b.

Armed with these descriptions we can define normalization, orthogonality, & completeness.

Normalization

$$\left.\begin{aligned}\langle+|+\rangle &= 1\\ \langle-|-\rangle &= 1\end{aligned}\right\}$$ inner products of same basis vectors equal 1

Orthogonality

$$\left.\begin{aligned}\langle+|-\rangle &= 0\\ \langle-|+\rangle &= 0\end{aligned}\right\}$$ inner products of different basis vectors equal 0

Completeness

$$|\psi\rangle = a|+\rangle + b|-\rangle \}$$ a general vector is described by the full basis.

Resulting Mathematical Outcomes

$$\langle + | \psi \rangle = \langle + | \left(a | + \rangle + b | - \rangle \right)$$

$$= \langle + | a | + \rangle + \langle + | b | - \rangle = a \overset{1}{\cancel{\langle + | + \rangle}} + b \overset{0}{\cancel{\langle + | - \rangle}}$$

$$\langle + | \psi \rangle = a \Leftarrow$$ This is the <u>probability amplitude</u>.

As we will see the square of this value is the probability of measuring $+\frac{\hbar}{2}$ for S_z.

$$\langle \psi | + \rangle = \langle + | a^* | + \rangle + \langle - | b^* | + \rangle = a^*$$

Note that $\langle + | \psi \rangle = \langle \psi | + \rangle^*$

Or more generally $\langle \psi | \phi \rangle = \langle \phi | \psi \rangle^*$

Probability of an observation

For our spin 1/2 system,

$$\left. \begin{aligned} P_{S_z = +\frac{\hbar}{2}} &= |\langle + | \psi \rangle|^2 \\ P_{S_z = -\frac{\hbar}{2}} &= |\langle - | \psi \rangle|^2 \end{aligned} \right\}$$ Amplitude^2 is the probability of obtaining a particular measurement.

(Postulate 4)

Example

In a particular experiment a beam of atoms is sent through a SG device. After a long time, the measurement appartus registers 5000 $+\hbar/2$ counts and 12000 $-\hbar/2$ counts. for S_z

— Determine the normalized state vector that describes an atom in this beam.

Solution: There were 19000 total counts

So,

$$P_{S_z=+\frac{\hbar}{2}} = \frac{5000}{19000} = \frac{5}{19} = |\langle +|\psi\rangle|^2$$

$$P_{S_z=-\frac{\hbar}{2}} = \frac{12000}{19000} = \frac{12}{19} = |\langle -|\psi\rangle|^2$$

a general state is given by,

$$|\psi\rangle = a|+\rangle + b|-\rangle$$

so

$$|\langle +|\psi\rangle|^2 = |a^* a| = a^2 = 5/19$$

$$|\langle -|\psi\rangle|^2 = |b^* b| = b^2 = 12/19$$

$$a = \sqrt{\frac{5}{19}} \qquad b = \sqrt{\frac{12}{19}}$$

Check normalization,

$$\langle \psi | \psi \rangle = a^2 + b^2 = \frac{5}{19} + \frac{12}{19} = 1 \checkmark$$

$$|\psi\rangle = \sqrt{\frac{5}{19}}\,|+\rangle + \sqrt{\frac{12}{19}}\,|-\rangle$$

- In fact, there could be a relative phase between the two, something like

$$\sqrt{\frac{5}{19}}\, e^{i\alpha}\,|+\rangle + \sqrt{\frac{12}{19}}\, e^{i\beta}\,|-\rangle$$

and overall phase doesn't matter, so, we could write this as,

$$\sqrt{\frac{5}{19}}\,|+\rangle + \sqrt{\frac{12}{19}}\, e^{ic}\,|-\rangle$$

We need observations of other measures to pin down this phase, so we ignore it for now.

Matrix Notation

One of the more powerful tools for doing QM is linear algebra. We can represent state vectors as column vector and build up a full formalism using Matrix Notation.

It is through this matrix notation that the S_z basis becomes more transparent.

$$|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$|+\rangle$

$|-\rangle$

"is represented by"

$$|-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Notice that the column vectors for $|+\rangle$ and $|-\rangle$ only have one nonzero entry. That is how you know $|+\rangle$ and $|-\rangle$ are "the basis vectors"

A general ket is described like this,

$$|\psi\rangle = a|+\rangle + b|-\rangle$$

$$|\psi\rangle \doteq a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle \doteq \begin{pmatrix} a \\ b \end{pmatrix} \Longleftarrow$$

With this representation we can also construct the corresponding bra

$$\langle\psi| = a^*\langle+| + b^*\langle-| \doteq a^*(1 \;\; 0) + b^*(0 \;\; 1)$$

$$\langle\psi| \doteq (a^* \;\; b^*)$$

The rules of matrix algebra then give us the same mathematical results as we have seen before,

$$\langle\psi|\psi\rangle = |a|^2 + |b|^2 = 1 \text{ (if normalized)}$$

$$\langle\psi|\psi\rangle \doteq (a^* \;\; b^*)\begin{pmatrix} a \\ b \end{pmatrix} = a^*a + b^*b$$

$$= |a|^2 + |b|^2 \checkmark$$

Other Spin States

We have described the state vectors in the S_z basis using kets & linear algebra,

$$|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We can use experimental results to derive the state vectors for the x & y projections of the spin angular momentum.

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We don't derive them here, but just state them. See McIntyre Section 1.2.2 & Ex 1.3 for derivations.

$$|+\rangle_x = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$|-\rangle_x = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$|+\rangle_y = \frac{1}{\sqrt{2}}|+\rangle + \frac{i}{\sqrt{2}}|-\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

$$|-\rangle_y = \frac{1}{\sqrt{2}}|+\rangle - \frac{i}{\sqrt{2}}|-\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$