Introduction to Signal Analysis

Now that we have seen we can use superposition to build solutions to the wave equation, we might ask can we use the concept of superposition to deal with more general (periodic) signals.

We've claimed that any $f(t)$ that is periodic can be written as,

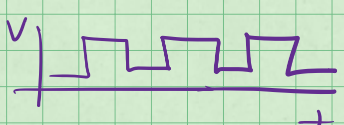
$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

where $n\omega_0$ represent harmonics of some base periodicity, $\omega_0 \Leftarrow$ typically the lowest observed frequency $\rightarrow$ longest periodic signal.

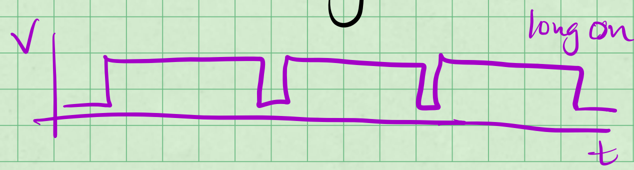
this is maybe best conducted via example

The Duty Cycle

A common signal in electronic systems is the duty cycle. A signal is turned off and on at some regular interval,



which can vary interval widths,



and can ramp up and down differently,

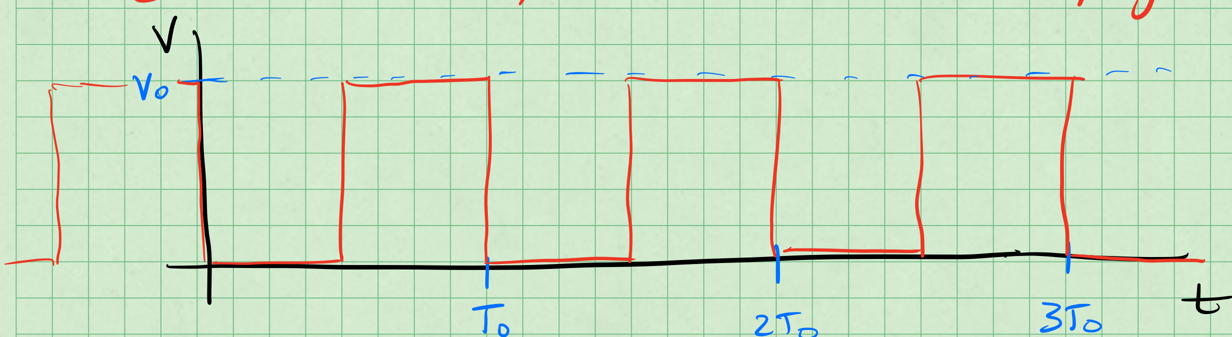
In all cases, we can use Fourier Decomposition

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$\boxed{\text{☆ later, we will use the fast fourier transform FFT to show how to get approximate } a_n\text{'s \& } b_n\text{'s.}}$

Example: The Square Wave

Consider the square wave (50% duty cycle),



Mathematically this signal is discontinuous,

$$V(t) = \begin{cases} 0 & 0 < t \leq T_0/2 \\ V_0 & \frac{T_0}{2} < t < T_0 \end{cases}$$

over one cycle

We will use Fourier's Trick to find a_n's & b_n's to approximate this duty cycle with continuous functions,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt \qquad n \neq 0$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt$$

using orthogonal functions to get coeffs.

Let's look at it first and compare to the model,

1. $V(t)$ is not symmetric so a_n's $= 0$. (follows sine symmetry)

2. $\frac{a_0}{2}$ is just the DC offset (average of signal)

   $\hookrightarrow a_0 = V_0$

Our simplified model

$$f(t) = \frac{V_0}{2} + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

All that's left is to find b_n,

$$b_n = \frac{2}{T_0}\int_0^{T_0} V(t)\sin(n\omega_0 t)\,dt \quad ; \quad \omega_0 = \frac{2\pi}{T_0}$$

$$V(t) = \begin{cases} 0 & 0 < t < T_0/2 \\ V_0 & T_0/2 < t < T_0 \end{cases}$$

Our integral simplifies to,

$$b_n = \frac{2}{T_0}\int_{T_0/2}^{T_0} V_0 \sin\left(n\frac{2\pi t}{T_0}\right)dt = \underbrace{\frac{2V_0}{T_0}\int_{\frac{T_0}{2}}^{T_0} \sin\left(\frac{2\pi n}{T_0}t\right)dt}$$

$$\left(\frac{T_0}{2\pi n}\right)\left(\cos(n\pi) - \underbrace{\cos(2n\pi)}_{1}\right)$$

$$b_n = \frac{2V_0}{T_0}\left(\frac{T_0}{2\pi n}\right)\left(\cos(n\pi) - 1\right)$$

$$\boxed{b_n = \frac{V_0}{n\pi}\left[\cos(n\pi) - 1\right]}$$

n	b_n
1	$-\frac{2V_0}{\pi}$
2	0
3	$-\frac{2V_0}{3\pi}$
4	0
5	$-\frac{2V_0}{5\pi}$
⋮	

$$V(t) = \frac{V_0}{2} + \sum_{n=1}^{\infty} \frac{V_0}{n\pi}\left[\cos(n\pi) - 1\right]\sin\left(\frac{2\pi n}{T_0}t\right)$$

!!!

But what does this look like?