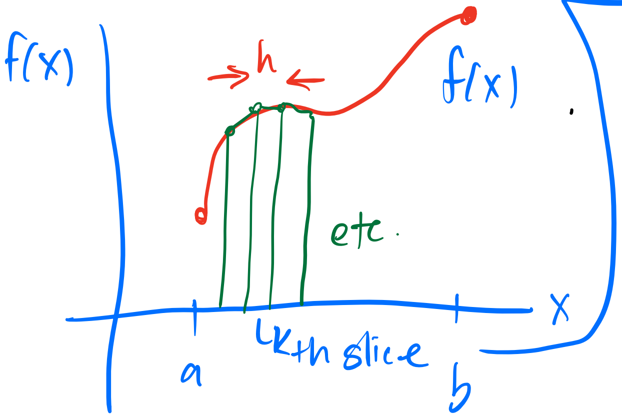
Integrating ODEs

Numerical Integration extends beyond the use to perform analytical integrals,

Trapezoidal Rule

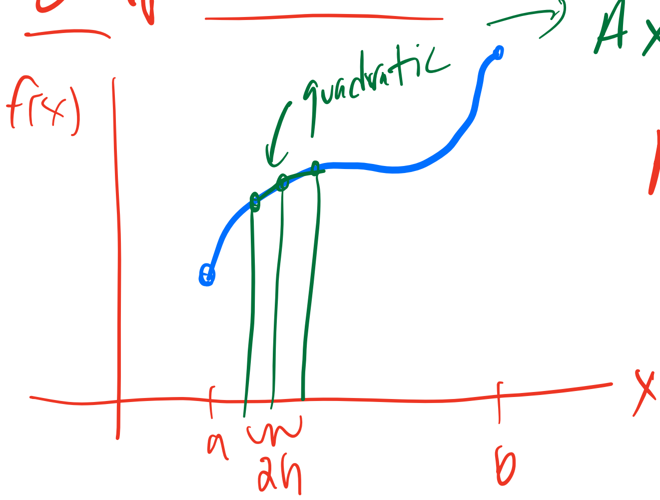


$$\text{Area}_k = \frac{1}{2}h\left[f(a+(k-1)h) + f(a+kh)\right]$$

$$I \simeq \sum_k A_k$$

$$I = h\left[\frac{1}{2}f(a) + \frac{1}{2}f(b) + \sum_{k=1}^{N-1} f(a+kh)\right]$$

Simpson's Rule



$$Ax^2 + Bx + C$$

$$A_k = \frac{1}{3}h\left(f(x_k) + 4f(x_{k+1}) + f(x_{k+2})\right]$$

$$I \simeq \sum_k A_k$$

$$I \simeq \frac{1}{3}h\left[f(a) + f(b) + 4\sum_{k\,\text{odd}}^{N-1} f(a+kh) + 2\sum_{k\,\text{even}}^{N-1} f(a+kh)\right]$$

We can use these to "integrate the equations of motion."

First Order Differential Equation

Let's see how this works for a 1st order ODE,

$$\frac{dx}{dt} = f(x,t) \quad \text{or} \quad \dot{x} = f(x,t)$$

Cool. Let's say we know where we are a time, t, and we want to predict (estimate) where we will be a short time, h, later.

The standard approach involves a Taylor expansion around t,

$$x(t+h) = x(t) + h\frac{dx}{dt} + \frac{1}{2}h^2\frac{d^2x}{dt^2} + \ldots$$

$$= x(t) + h\frac{dx}{dt} + O(h^2)$$

order h^2 and above terms.

So our linear (in h) approx gives,

$$x(t+h) = x(t) + h\frac{dx}{dt} \quad \text{or,}$$

$$\boxed{x(t+h) = x(t) + h f(x,t)}$$

Euler integral.

Great! So for first order ODEs we can use this!

2nd Order ODEs?

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right) \quad \text{or} \quad \ddot{x} = f(x, \dot{x}, t)$$

we make two first order ODEs,

let $v = \frac{dx}{dt}$ then $\frac{dv}{dt} = \frac{d^2x}{dt}$

So that,

$$\frac{dv}{dt} = f(x, v, t) \quad \text{and} \quad \frac{dx}{dt} = v$$

then like before,

$$v(t+h) = v(t) + h f(x,v,t)$$
$$x(t+h) = x(t) + h v(t)$$

Euler for 2nd order

!! This approach does not conserve energy ↑
It will have big issues long term or for oscillations.

It is corrected by Cromer (1963)

$$v(t+h) = v(t) + h f(x, v, t)$$
$$x(t+h) = x(t) + h \underbrace{v(t+h)}$$

take the prediction from prior step & use here

Euler-Cromer Integration 2nd Order

$$v(t+\Delta t) = v(t) + \frac{F(x,v,t)}{m} \Delta t$$

← net force

$\vec{F}_{net} = m\vec{a}$

$$x(t+\Delta t) = x(t) + v(t+\Delta t)\Delta t$$