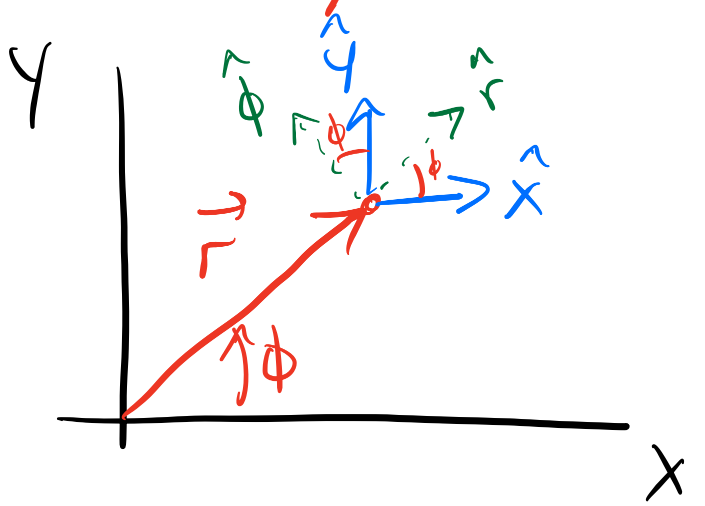
Deriving Newton's 2nd Law in Plane Polar

1. Draw $\vec{r}$
2. compute $d\vec{r}/dt = \vec{v}$
3. compute $d^2\vec{r}/dt^2 = \vec{a}$
4. investigate $\vec{F} = m\vec{a}$

①



$\vec{r} = r\hat{r}$ where $r = |\vec{r}|$

②

$$\frac{d\vec{r}}{dt} = \underbrace{\frac{dr}{dt}}_{\dot{r}}\hat{r} + r\underbrace{\frac{d\hat{r}}{dt}}_{?}$$

$$\hat{r} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

do in cartesian

↳ $$\frac{d\hat{r}}{dt} = -\dot{\phi}\sin\phi\,\hat{x} + \dot{\phi}\cos\phi\,\hat{y} = \dot{\phi}\hat{\phi}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

(3)

$$\vec{a} = \frac{d\vec{v}}{dt} = \underbrace{\frac{d\dot{r}}{dt}}_{\ddot{r}}\hat{r} + \dot{r}\underbrace{\frac{d\hat{r}}{dt}}_{\dot{\phi}\hat{\phi}} + \underbrace{\frac{dr}{dt}}_{\dot{r}}\dot{\phi}\hat{\phi} + r\underbrace{\frac{d\dot{\phi}}{dt}}_{\ddot{\phi}}\hat{\phi} + r\dot{\phi}\underbrace{\frac{d\hat{\phi}}{dt}}_{?}$$

$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\cos\phi\,\hat{x} - \dot{\phi}\sin\phi\,\hat{y} = -\dot{\phi}\hat{r}$$

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\phi}\hat{\phi} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}(-\dot{\phi}\hat{r})$$

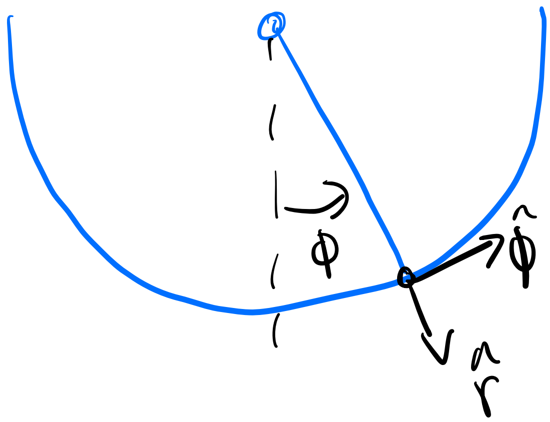
$$\vec{a} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi}$$

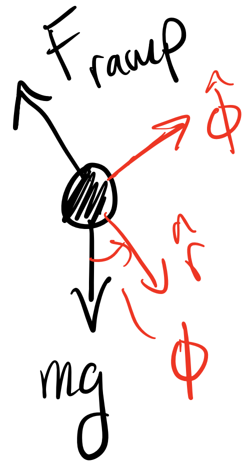
(4)

$$\vec{F}_{net} = m\vec{a} \Rightarrow \vec{F}_r + \vec{F}_\phi = m(\vec{a}_r + \vec{a}_\phi)$$

$$F_r = m(\ddot{r} - r\dot{\phi}^2) \qquad F_\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$$

Skateboard Example





$$\sum F_r = ma_r = -F_{ramp} + mg\cos\phi = m(\ddot{r} - r\dot{\phi}^2)$$

Note $r = R$ so $\ddot{r} = 0$,

$$-F_{ramp} + mg\cos\phi = -mR\dot{\phi}^2$$

$$\sum F_\phi = ma_\phi = -mg\sin\phi = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$$

Note: $r = R$ $\dot{r} = 0$

$$-mg\sin\phi = mR\ddot{\phi}$$

or $$\boxed{\ddot{\phi} = -\frac{g}{R}\sin\phi}$$

Assume small osc.

$\sin\phi \approx \phi$

$$\ddot{\phi} = -\frac{g}{R}\phi \implies \ddot{x} = -\omega^2 x\,?$$

$$\omega^2 = g/R \quad \text{so} \quad \omega = \sqrt{g/R}$$

$$\phi(t) = A\cos(\omega t) + B\sin(\omega t) \quad \text{general soln.}$$