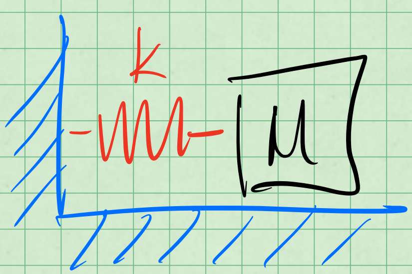
Coupled Oscillations

Let's remind ourselves of single SHO results.



EOM

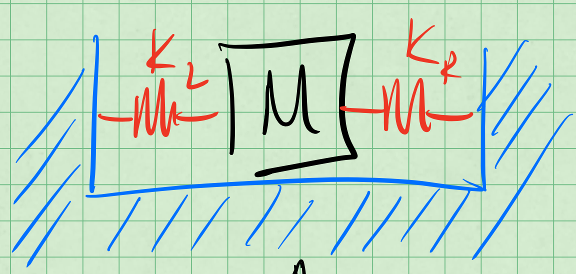
$$m\ddot{x} = -kx$$

$$\ddot{x} = -\frac{k}{m}x = -\omega^2 x$$

$$\omega = +\sqrt{k/m}$$

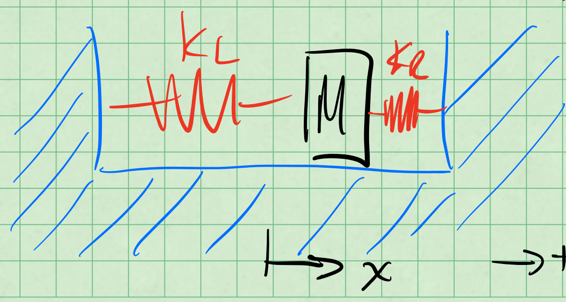
$$x(t) = A\cos(\omega t + \phi)$$ ⇐ we've used this many times

Let's add another spring,

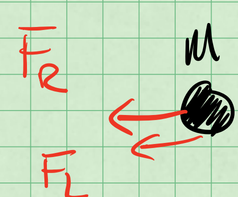


k_L & k_R different spring constants

↑ concerned with displacement from equilibrium



→ +x direction



Both try to return mass.

EOM: $M\ddot{x} = -k_L x - k_R x$ ②

$$M\ddot{x} = -(k_L + k_R)x$$

so

$$\boxed{\ddot{x} = -\left(\frac{k_L + k_R}{M}\right)x} = -\omega^2 x$$

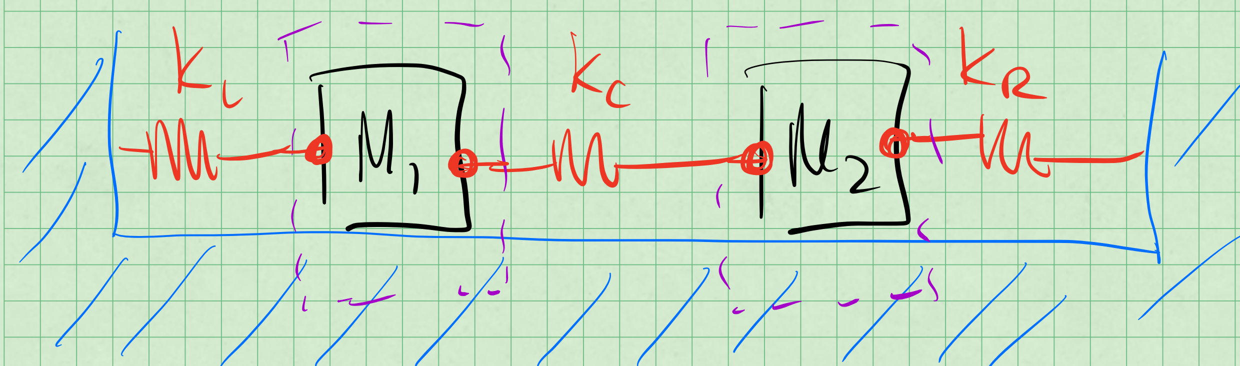
where

$$\omega = +\sqrt{\frac{k_L + k_R}{M}} \qquad k_L + k_R = k_{eff}$$

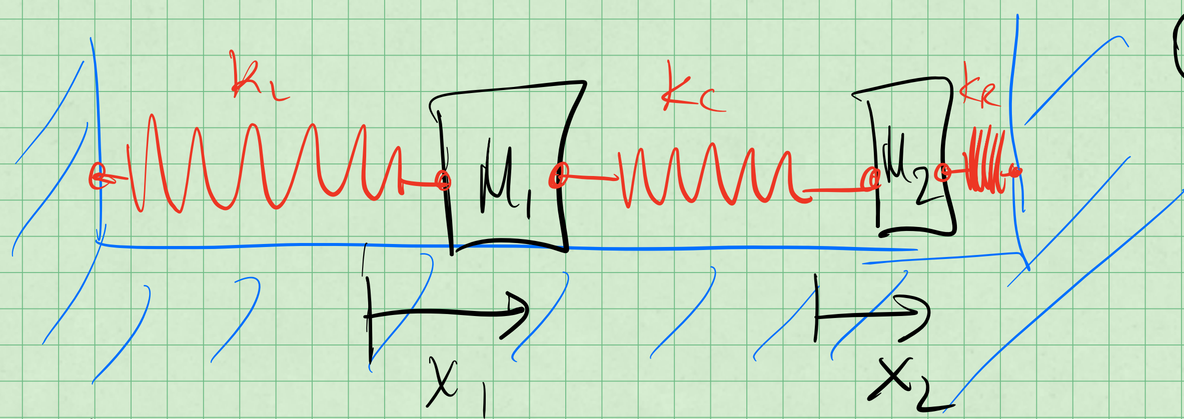
So no difference from first case except change in ω,

$$x(t) = A\cos\left(\sqrt{\frac{k_{eff}}{M}}\,t + \phi\right)$$

Lets Bring In a second Mass,



Note: we only have two interactions per mass. The dotted lines show this.



Again only concerned about displacement from equilibrium. We get 2 linear 2nd order Diffy Qs,

EOMs:

$$m\ddot{x}_1 = -k_L x_1 + k_C(x_2 - x_1)$$

$$m\ddot{x}_2 = -k_R x_2 - k_C(x_2 - x_1)$$

Newton's Third Law pairs!

and thus,

$$\boxed{\begin{aligned}\ddot{x}_1 &= -\frac{(k_L+k_C)}{m_1}x_1 + \frac{k_C}{m_1}x_2 \\ \ddot{x}_2 &= +\frac{k_C}{m_2}x_1 - \frac{(k_R+k_C)}{m_2}x_2\end{aligned}}$$

We have 2 coupled 2nd order linear Diffy Qs. We have many tools to solve

Let's make this model simpler where

$k_L = k_C = k_R = k$ and $M_1 = M_2 = M$

- same masses and springs
- good model in QM, Stat Mech, Waves, & CMP solids.

New EOM (same m & k's)

$$\ddot{x}_1 = -\frac{2k}{m} x_1 + \frac{k}{m} x_2$$

$$\ddot{x}_2 = \frac{k}{m} x_1 - \frac{2k}{m} x_2$$

How to solve?

Let's do this in 2 ways,

1) linear change of variable to decouple the eqns.
2) normal mode analysis

Change of Variables

We are going to make a change of variable specifically,

$$x_{cm} = \underbrace{\frac{x_1 + x_2}{2}}_{\text{center of mass coord}} \qquad x_{dis} = \underbrace{\frac{x_1 - x_2}{2}}_{\text{seperation from equil.}}$$

These are linear transforms b/c (5)

$$\dot{x}_{cm} = \frac{dx_{cm}}{dt} = \frac{1}{2}\left(\frac{dx_1}{dt} + \frac{dx_2}{dt}\right)$$ no cross terms!

$$\ddot{x}_{cm} = \frac{d^2x_{cm}}{dt} = \frac{1}{2}(\ddot{x}_1 + \ddot{x}_2)$$

Lets eliminate x_1 & x_2 for x_{cm} & x_{dis}

$$\ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2 \qquad \ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2$$

$$\frac{\ddot{x}_1 + \ddot{x}_2}{2} = \left(\frac{-k}{m}x_1 + \frac{k}{2m}x_2 + \frac{k}{2m}x_1 - \frac{k}{m}x_2\right)$$

$$\frac{\ddot{x}_1 + \ddot{x}_2}{2} = -\frac{k}{2m}(x_1 + x_2)$$ so that,

$$\boxed{\ddot{x}_{cm} = -\frac{k}{m}x_{cm}}$$

$$\frac{\ddot{x}_1 - \ddot{x}_2}{2} = \left(-\frac{k}{m}x_1 + \frac{k}{2m}x_2 - \frac{k}{2m}x_1 + \frac{k}{m}x_2\right)$$

$$\frac{\ddot{x}_1 - \ddot{x}_2}{2} = -\frac{3k}{2m}(x_1 - x_2)$$ so that

$$\boxed{\ddot{x}_{dis} = -\frac{3k}{m}x_{dis}}$$

So we found SHO! (6)

$$\ddot{x}_{cm} = -\omega_{cm}^2 x_{cm} \quad \text{where} \quad \omega_{cm} = \sqrt{k/m}$$

$$\ddot{x}_{dis} = -\omega_{dis}^2 x_{dis} \quad \text{where} \quad \omega_{dis} = \sqrt{\frac{3k}{m}}$$

and thus,

$$x_{cm}(t) = A\cos(\omega_{cm}t + \phi_{cm})$$

$$x_{dis}(t) = B\cos(\omega_{dis}t + \phi_{dis})$$

are general solutions

Given initial conditions $x_1(t=0)$, $\dot{x}_1(t=0)$, $x_2(t=0)$, $\dot{x}_2(t=0)$

we can find particular solutions.
and we can find $x_1(t)$ & $x_2(t)$
by simply adding/subtracting solutions,

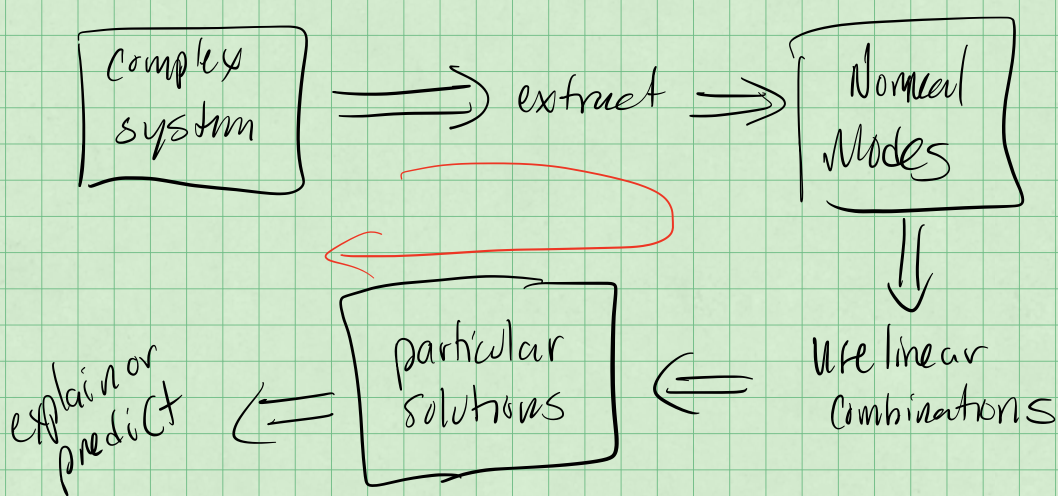
$$x_{cm} = \frac{x_1 + x_2}{2} \qquad x_{dis} = \frac{x_1 - x_2}{2}$$

$$\boxed{\begin{aligned} x_1(t) &= x_{cm}(t) + x_{dis}(t) \\ x_2(t) &= x_{cm}(t) - x_{dis}(t) \end{aligned}}$$

Normal Modes

The approach we took above works well for 2 bodies because the center of mass is a natural choice for change of variable. But with N bodies, how do we choose linear transformations? It's hard! So let's look at an approach that works for many linear problems: Normal Modes.

Normal Modes can be thought of as natural often simple behaviors. They have unique oscillation frequencies. *sometimes they can be degenerate (same ω, diff mode)



Process of using normal modes

We assume a type of solution here, (8)

$x(t) = A\cos(\omega t + \phi)$ or $x(t) = Ce^{i\omega t}$

oscillating solutions with unknown constant coeffs

Reminder: Diffy Q.

$$\ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2$$

$$\ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2$$

plugging in

$x_1 = C_1 e^{i\omega t}$

$x_2 = C_2 e^{i\omega t}$ gives,

1 2 3

$$\left.\begin{aligned} A)\ -\underline{\omega}^2 \underline{C_1} &= -\frac{2k}{m}\underline{C_1} + \frac{k}{m}\underline{C_2} \\ B)\ -\underline{\omega}^2 \underline{C_2} &= \frac{k}{m}\underline{C_1} - \frac{2k}{m}\underline{C_2} \end{aligned}\right]$$

three unknowns!

two eqns.

This ok b/c C_1 or C_2 sets the other (energy limited; only 1 value of total energy possible)

Add A & B

$$-\omega^2 C_1 - \omega^2 C_2 = -\frac{2kC_1}{m} + \frac{k}{m}C_2 + \frac{k}{m}C_1 - \frac{2k}{m}C_2$$

$$-\omega^2\cancel{(C_1+C_2)} = -\left(\frac{k}{m}\right)\cancel{(C_1+C_2)}$$

C_1 & $C_2 \neq 0$ b/c amplitudes

$$\boxed{\omega^2 = \frac{k}{m}}$$

Subtract A & B

$$-\omega^2(c_1-c_2) = -\frac{2k}{m}c_1 + \frac{k}{m}c_2 - \frac{k}{m}c_1 + \frac{2k}{m}c_2$$

$$-\omega^2(\cancel{c_1-c_2}) = -\frac{3k}{m}(\cancel{c_1-c_2}) \qquad \boxed{\omega^2 = \frac{3k}{m}}$$

Hold up!

These transformations and our normal mode analysis gave us the same ω's!

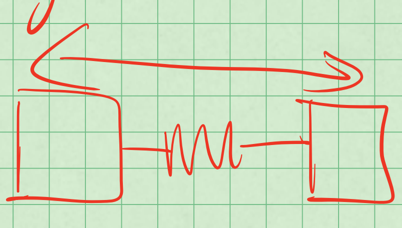
That b/c our transformations were moving us to normal modes!

] this will not be the case always

we picked a good transform.

Mode 1: $\omega^2 = k/m$

a basic oscillation w/ one spring.

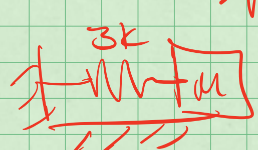


whole system oscillates

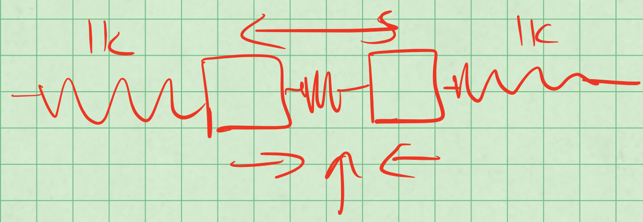
no inner spring compression

Mode 2: $\omega^2 = \frac{3k}{m} \leftarrow$ higher frequency oscillation

like a spring with 3k spring constant.



we compressed the middle spring!



Mode 2

oscillating in opp. directions

2k! both compress!

OK this is cool but can we generalize this approach? Yes!

Consider writing our EOMs as a matrix equation.

$$\ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2$$

$$\ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2$$

let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\ddot{\vec{x}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}$

$$\ddot{\vec{x}} = \underline{\underline{A}}\,\vec{x}$$

$\underline{\underline{A}}$ matrix of coefficients

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -2k/m & k/m \\ k/m & -2k/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \underline{\underline{A}} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$$

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We assume a solution again $x(t) = Ce^{i\omega t}$

so, this produces this eigenvalue problem

$$\ddot{\vec{x}} = \boxed{A\vec{x} = -\omega^2\vec{x}}$$

Eigenvalue Problem

note the + sign b/c $\omega^2 > 0$

$$(\underline{\underline{A}} + \underline{\underline{I}}\,\omega^2)\vec{x} = 0$$

$\det|A + I\omega^2| = 0$ gives ω eigenvalues

$$\det \begin{vmatrix} -\frac{2k}{m} + \omega^2 & +\frac{k}{m} \\ +\frac{k}{m} & -\frac{2k}{m} + \omega^2 \end{vmatrix} = 0$$

$$\left(-\frac{2k}{m} + \omega^2\right)^2 - \frac{k^2}{m^2} = 0$$

$$-\frac{2k}{m} + \omega^2 = \pm \frac{k}{m}$$

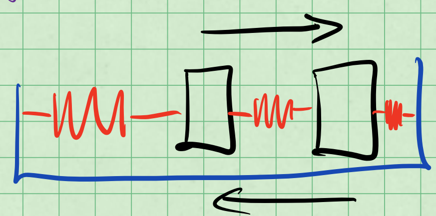
$$\omega^2 = \pm\frac{k}{m} + \frac{2k}{m} \Rightarrow \frac{k}{m} \text{ and } \frac{3k}{m}$$

$$\boxed{\omega_1 = +\sqrt{\frac{k}{m}} \qquad \omega_2 = +\sqrt{\frac{3k}{m}}}$$

Same two modes!!

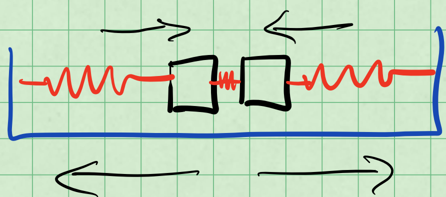
Ok but how do we know these modes correspond to:

Mode 1:



move together
no central spring compression

Mode 2:



compress central spring then push apart

Let's go back to $x_1(t)$ & $x_2(t)$

for mode 1 $x_1(t) = x_2(t)$
that is, the oscillations follow exactly.

for mode 2 $x_1(t) = -x_2(t)$
that is, the oscillation are precisely $180°$ out of phase.