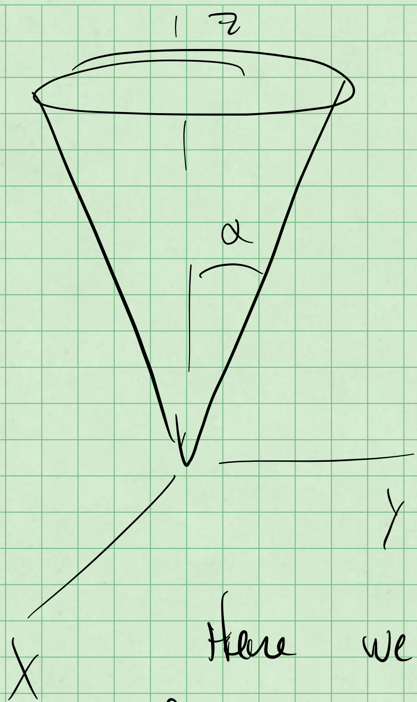


$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$

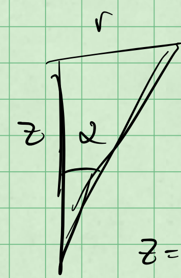
$U = mgz$

constraint?

$f(r,z) = 0$

$\parallel$

$z - r\cot\alpha = 0$



$\tan\alpha = \frac{r}{z}$

$z = r\cot\alpha$

Here we aim to find the generalized forces, $Q_j = \lambda \frac{\partial f}{\partial q_j}$ using the method of Lagrange Multipliers.

Instead of putting in the constraint immediately & finding 2 EOMs, we instead solve for all 3 using the modified Euler Lagrange Eqns.

$$\boxed{\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) + \underbrace{\lambda \frac{\partial f}{\partial q}}_{\text{constraint eqn!}} = 0}$$

$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz \quad f = z - r\cot\alpha = 0$

r EoM

$$\frac{d\mathcal{L}}{dr} - \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{r}}\right) + \lambda \frac{df}{dr} = 0$$

$$mr\dot{\theta}^2 - \frac{d}{dt}\left(m\dot{r}\right) + \lambda\left(-\cot\alpha\right) = 0$$

$$\boxed{mr\dot{\theta}^2 - m\ddot{r} - \lambda\cot\alpha = 0}$$

θ EoM

$$\frac{d\mathcal{L}}{d\theta} - \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}}\right) + \lambda \frac{df}{d\theta} = 0$$

$$0 - \frac{d}{dt}\left(mr^2\dot{\theta}\right) + \lambda(0) = 0$$

$$\boxed{\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0}$$

z EoM

$$\frac{d\mathcal{L}}{dz} - \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{z}}\right) + \lambda \frac{df}{dz} = 0$$

$$-mg - \frac{d}{dt}\left(m\dot{z}\right) + \lambda = 0$$

$$\boxed{m\ddot{z} + mg - \lambda = 0}$$

We have 4 unknowns, r, θ, z & λ. But we have 4 eqns. 3 EoMs + 1 constraint Eqn.

$$m r \dot{\theta}^2 - m\ddot{r} - \lambda \cot\alpha = 0 \quad (1)$$

$$\frac{d}{dt}\left(m r^2 \dot{\theta}\right) = 0 \quad (2)$$

$$m\ddot{z} + mg - \lambda = 0 \quad (3)$$

$$z = r\cot\alpha \quad (4)$$

$$\dot{z} = \dot{r}\cot\alpha$$

$$\ddot{z} = \ddot{r}\cot\alpha$$

Our goal is to find λ b/c $Q_j = \lambda \frac{df}{dq_j}$

$$\lambda = m(\ddot{z} + g) = m(\ddot{r}\cot\alpha + g)$$

So, $$\ddot{r} = \left(\frac{\lambda}{m} - g\right)\tan\alpha$$

or

$$m r \dot{\theta}^2 - m\left(\frac{\lambda}{m} - g\right)\tan\alpha - \lambda\cot\alpha = 0$$

$$m r \dot{\theta}^2 - \lambda\tan\alpha + g\tan\alpha - \lambda\cot\alpha = 0$$

$$m r \dot{\theta}^2 - \lambda(\tan\alpha + \cot\alpha) + g\tan\alpha = 0$$

$$\lambda = \left(\frac{m r \dot{\theta}^2 + g\tan\alpha}{\tan\alpha + \cot\alpha}\right)$$

this is in terms of dynamical variables r & $\dot{\theta}$ so we can be done

α is always less than $\pi/2$

so $\tan\alpha > 0$ and $\cot\alpha > 0$

so $\lambda > 0$

$Q_r = \lambda \frac{\partial f}{\partial r} = -\lambda \cot\alpha$ points inward! ✓

$Q_\theta = \lambda \frac{\partial f}{\partial \theta} = 0$ no torque! L_z conserved.

$Q_z = \lambda \frac{\partial f}{\partial z} = +\lambda$ points upward! ✓