- So, we generated a solution to Laplace's equation for situations where we make use of Cartesian coordinates.

Sometimes our geometry for the problem lends it self to some other ~~g~~ coordinate system. For example a common problem will involve spherical geometries → so $X(x)\,Y(y)\,Z(z)$ will work, but the solution will be a terrible mess → just nasty equations.

Clicker Question: Can we try $R(r)\,\Theta(\theta)\,\Phi(\varphi)$?

* Turns out we can use separation of variables in other coordinate systems, such as spherical.

In this class we limit ourselves to azimuthally symmetric problems. That is $V(r,\theta,\varphi) = V(r,\theta)$ only!

So our Ansatz will be $V = R(r)\,\Theta(\theta)$

Let's plug it into $\nabla^2 V$ in spherical coordinates and see what we get.

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \underbrace{0}_{\text{no }\varphi} = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}(R\Theta)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{d}{d\theta}(R\Theta)\right) = 0$$

$$\Theta\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\right] + R\left[\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right] = 0$$

Let's Divide by $V = R\Theta$ again,

$$\frac{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}{R(r)} + \frac{\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}{\Theta(\theta)} = 0$$

Cleanup by multiplying by r^2,

$$\underbrace{\frac{\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}{R(r)}}_{= C_1} + \underbrace{\frac{\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}{\Theta(\theta)}}_{= C_2} = 0$$

where $C_1 + C_2 = 0$ as we had before because each term depends only on r or θ.

So we have once again turned our PDE problem into a pair of 2nd order ODEs!
We need to find their general solutions & then we are left with just matching the BC's!

★ So which one (C_1 or C_2) is positive?
We won't prove this, but we know that $\Theta(\theta)$ <u>must not</u> blow up → we can't all $V \to \infty$ at finite r's. So this condition forces C_2 to have two features:

① $C_2 < 0$ and

② $C_2 = -l(l+1)$ where $l \geq 0$ and <u>an integer</u>

so, $C_1 = +l(l+1)$
$C_2 = -l(l+1)$

ODEs with these constants are solvable!

for the equation for $R(r)$, we get,

$$\frac{d}{dr}\left(r^2 \frac{dR(r)}{dr}\right) = \ell(\ell+1)R(r)$$

The general solution to this differential equation (which you can check!) is:

$$\boxed{R(r) = Ar^{\ell} + B/r^{\ell+1}}$$

where A & B are constants determined by the BCs.

The angular differential Equation is a little nastier,

$$\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta(\theta)}{d\theta}\right) = -\ell(\ell+1)\sin\theta\, \Theta(\theta)$$

yuck! But the solutions turn out to be not so bad! Suprisingly

Take $\ell = 0$,

$$\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta(\theta)}{d\theta}\right) = 0$$

is solved by $\Theta(\theta) = \text{constant}!$ Because our solution is multiplicative $V = R\Theta$, we can absorb any arbitrary constants from $\Theta(\theta)$ into A & B for $R(r)$. So we choose $\Theta_0(\theta) = 1$

* the subscript indicates we solved this for $\ell = 0$.

For $\ell = 1$,

The solution is $\Theta_1(\theta) = \cos\theta$, let's check it!

$$\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}(\cos\theta)\right) = -(1)(2)\cos\theta\sin\theta \quad \checkmark$$

It turns out the differential Equation

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta(\theta)}{d\theta}\right) = -\ell(\ell+1)\sin\theta\,\Theta(\theta)$$

is solved by the Legendre Polynomials for $\cos\theta$

$$\boxed{\Theta_\ell(\theta) = P_\ell(\cos\theta)}$$

$$P_0(\cos\theta) = 1 \qquad P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2} \qquad P_3(\cos\theta) = \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta$$

So our general solution for azimuthally symmetric problems is,

$$V_\ell(r,\theta) = R_\ell(r)\,\Theta_\ell(\theta) = \left(Ar^\ell + \frac{B}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

This solves $\nabla^2 V_\ell = 0$ and it is true for any ℓ so superposition allows us to write the <u>completely</u> <u>general</u> <u>solution</u>:

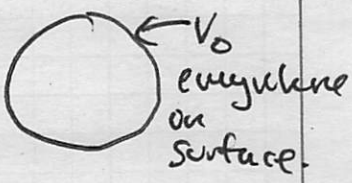
$$\boxed{V(r,\theta) = \sum_{\ell=0}^{\infty}\left(Ar^\ell + \frac{B}{r^{\ell+1}}\right)P_\ell(\cos\theta)}$$

<u>Here's the rub:</u>

- In a spherical situation with azimuthal symmetry, the equation above is the solution.
- You will have to match the BCs to find which A_ℓ's & B_ℓ's survive and what form they take.
- You build up your solution in this way.
- If you can do that, uniqueness guarentees you've found the solution!

Consider a spherical shell, doesn't need to be metal.

You know $V = V_0$ on the surface



Given $V(r,\theta) = \sum_l (Ar^l + B/r^{l+1}) P_l(\cos\theta)$,

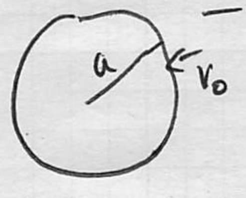
which terms survive for V inside the spherical shell?

<u>Clicker Question:</u> $V = V_0$ on surface! A_l's & B_l's?

So we can use our boundary conditions to generate the solution by considering which terms need to survive.

<u>Example:</u> Outside that sphere w/ V_0 on surface.

- All A_l's must $= 0$ b/c blow up @ $r \to \infty$.
- Solution must be spherically symmetric so, B_l's $= 0$ for $l \geq 1$. only $P_0(\cos\theta) = 1$ survives,

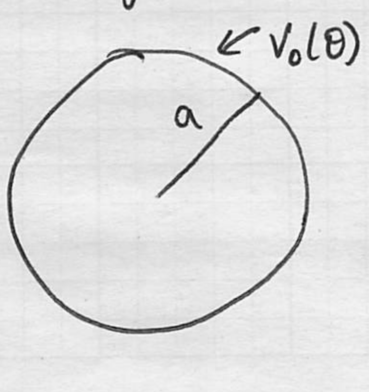


$$V(r,\theta) = \sum_l (A_l r^l + B_l/r^{l+1}) P_l(\cos\theta)$$

$$= \frac{B_0}{r} P_0(\cos\theta) = \frac{B_0}{r}$$

$$V(a,\theta) = V_0 = \frac{B_0}{a} \qquad B_0 = V_0 a \quad \text{so, } V(r) = \frac{V_0 a}{r}$$

<u>More general Example:</u> the boundary now has a potential that depends on θ,



$V(a,\theta) = V_0(\theta)$ given

We want $V(r,\theta)$ outside again.

$$V(r,\theta) = \sum_{l} \left(A_l r^l + B/_{r^{l+1}} \right) P_l(\cos\theta)$$

First, as $r \to \infty$, $V \to 0$ so all A_l's are zero!

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Our second Boundary condition demands,

$$V(a,\theta) = V_0(\theta) = \sum_{l=0}^{\infty} B_l/_{a^{l+1}} P_l(\cos\theta)$$ B_l's are only unknown.

As it turns out the Legendre polynomials form an orthogonal set of functions!

★ We can apply Fourier's trick to find the B_l's!

$$\int_{0}^{+1} P_l(u) P_m(u)\, du = \begin{cases} 0 & l \neq m \\ \frac{2}{2l+1} & l = m \end{cases}$$

<u>Clicker Question</u>: transform to $P_l(\cos\theta)$?

Turns out the integral we want is,

$$\int_0^{\pi} P_l(\cos\theta) P_m(\cos\theta) \sin\theta\, d\theta = \begin{cases} 0 & l \neq m \\ \frac{2}{2l+1} & l = m \end{cases}$$

So to find the B_l's in our problem, we multiply both sides by $P_m(\cos\theta)\sin\theta$ and integrate from 0 to π. All terms vanish except $l=m$!

So,
$$\frac{B_l}{a^{l+1}} \frac{2}{2l+1} = \int_0^{\pi} P_l(\cos\theta) V_0(\theta) \sin\theta\, d\theta$$

this gives all the B_l's!

If we want V inside the sphere given $V_0(\theta)$ on the sphere, the argument is similar, except $B_\ell = 0$ now ($V(0)$ must be finite!)

$$V_0(\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \quad \text{with}$$

$$A_\ell a^\ell \frac{2}{2\ell+1} = \int_0^\pi P_\ell(\cos\theta)\, V_0(\theta) \sin\theta\, d\theta$$

<u>Comments on Matching Boundary Conditions</u>

We have a method for solving for $V(r,\theta)$ in general by matching our BC's. As long as there's nothing funny happening, we typically need to just find the A_ℓ's & B_ℓ's given $V_0(\theta)$ on the surface of the sphere. (Usually all A_ℓ's vanish outside and all B_ℓ's vanish inside)

The integrals we have to solve look really nasty,

$$\int_0^\pi P_\ell(\cos\theta)\, V_0(\theta) \sin\theta\, d\theta$$

And in general, if $V_0(\theta)$ is nasty, we could be in for solving <u>a lot</u> of integrals. But often the orthogonality of $P_\ell(\cos\theta)$ will get us out of trouble.

$$\int_0^\pi P_\ell(\cos\theta) P_m(\cos\theta) \sin\theta\, d\theta = 0 \quad \ell \neq m!$$

So if $V_0(\theta)$ is a single Legendre polynomial or even a sum of them we might be able to just write down the solution.

For example,

if $V_0(\theta) = V_0$, $V_0(\theta) = V_0 P_0(\cos\theta)$ so we only need to solve one integral with $\ell = 0$ as the rest all vanish.

$P_0 = 1$ $\quad P_1 = \cos\theta$ $\quad P_2 = \frac{3}{2}\cos^2 - \frac{1}{2}$

Clicker Question: $V(R,\theta) = V_0(1+\cos^2\theta)$? inside & out.

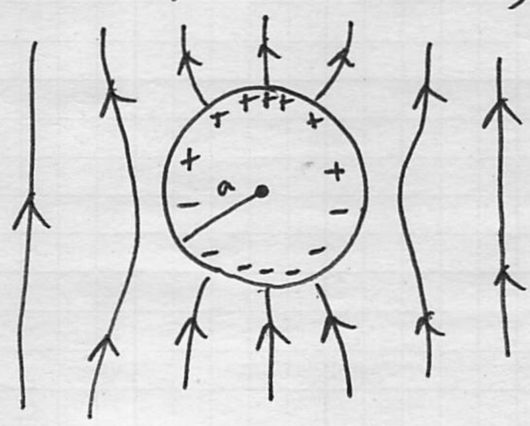
if $V_0(\theta) = \sin^2\theta$, I know I can write this as $V_0(\theta) = 1 - \cos^2\theta$ with $P_0 = 1$ and $P_2 = \frac{3}{2}\cos^2\theta - \frac{1}{2}$,

$$V_0(\theta) = -\frac{2}{3}P_2(\cos\theta) - \frac{2}{3}P_0$$

So only $\ell = 2$ & $\ell = 0$ will survive!

Detailed Classic Example:

Let's place a metal sphere of radius a into an existing, external, uniform field, $\vec{E} = E_0\hat{z}$.



We see the metal sphere will become polarized, generating a polarization field, which superposes with the external field producing a complicated field. Our job: Find $V(r,\theta)$ everywhere.

Every on and inside the metal sphere is at the same potential — metals are equipotential surfaces.

So,

$$V(r<a, \theta) = V_0$$

- We are free to pick where $V=0$, so let's pick $V(r=0)=0$ so that $V_0 = 0$ as the whole sphere is at the same potential.
- Outside the sphere $V \not\to 0$ as $r \to \infty$ because we chosen $V_0 = 0$. Can we determine what V is really far away? Sure, Let's use the field.

With $\vec{E}_{ext} = E_0 \hat{z}$, when we are really far from the sphere, the polarization field has died off. This leaves the total field to be just the external field,

$$\vec{E}_{tot} = E_0 \hat{z} \quad \text{as } r \to \infty$$

$$-\nabla V = E_0 \hat{z} \longrightarrow V(r\to\infty, \theta) = -E_0 z$$

* We have no constant because we expect $V=0$ far from the sphere $(x\to\infty, y\to\infty)$ on the plane $z=0$.

Our general solution for $V(r,\theta)$ is,

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

We solve outside first $(r>a)$ where

$$V(r\to\infty, \theta) \to -E_0 z = -E_0 r\cos\theta = -E_0 r P_1(\cos\theta)$$

So as $r \to \infty$,

$$-E_0 r P_1(\cos\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)\Bigg|_{r\to\infty} P_l(\cos\theta)$$

the B_l terms don't contribute as $r \to \infty$ as they die off, so we can only make sense of the A_l terms.

$$-E_0 r P_1(\cos\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

The boundary condition at $r \to \infty$ is purely written in terms of P_1, so only $l=1$ could contribute.

$$A_1 r P_1(\cos\theta) = -E_0 r P_1(\cos\theta) \Rightarrow A_1 = -E_0$$

All the other A_l's must vanish!

$$V(r,\theta) = A_1 r P_1(\cos\theta) + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Let's match at $r=a$, where we picked $V=0$.

$$V(a,\theta) = 0 = A_1 a P_1(\cos\theta) + \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos\theta)$$

the only way this could be zero is if all the B_l's are zero except $l=1$. Why?

B/c otherwise we are left with uncancelled functions of $\cos\theta$!

So,

$$V(a,\theta) = 0 = A_1 a P_1(\cos\theta) + \frac{B_1}{a^2} P_1(\cos\theta)$$

so that

$$A_1 a = -B_1/a^2 \quad \text{or} \quad B_1 = -A_1 a^3$$

Thus,

$$V(r,\theta) = A_1 r P_1(\cos\theta) - \frac{A_1 a^3}{r^2} P_1(\cos\theta) \quad \text{with } A_1 = -E_0$$

We find that for $r>a$,

$$V(r,\theta) = E_0 P_1(\cos\theta)\left(\frac{a^3}{r^2} - r\right)$$

$$= \underbrace{-E_0 r\cos\theta}_{\text{due to the external field}} + \underbrace{\frac{a^3}{r^2} E_0 \cos\theta}_{\text{due to the polarization field}}$$

and,

$V = 0$ for $r<a$

- Setting V is one way of determining boundary conditions, but we could have similarly specified the charge density $\sigma(a,\theta)$.

The approach is similar, now the boundary condition is on the derivative of the potential,

$E^{\perp}_{out} - E^{\perp}_{in} = \sigma/\epsilon_0$ and $E^{\perp} = \frac{-\partial V}{\partial r}$ for a sphere

so,

$$\left.\frac{\partial V}{\partial r}\right|_{r=a+\varepsilon} - \left.\frac{\partial V}{\partial r}\right|_{r=a-\varepsilon} = -\sigma/\epsilon_0$$

given

is the boundary condition

So using seperation of variables, you treat $r>a$ and $r<a$ cases separately then match them using the boundary condition above.

In the previous example we can use the discountinuity in $E^{\perp}$ to find how the charge is distributed on the surface of the sphere.

for $r \leq a$ $\quad V = 0$ $\quad$ so $\left.\frac{\partial V}{\partial r}\right|_{inside} = 0$ $\left(\begin{matrix}\text{also}\\ E=0\end{matrix}\right)$

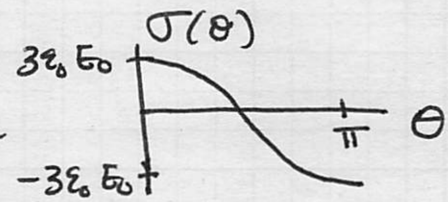
$$\left.\frac{\partial V}{\partial r}\right|_{just\,outside} = \frac{\partial}{\partial r}\left(-E_0 r\cos\theta + \frac{a^3}{r^2}E_0\cos\theta\right)$$

$$= -E_0\cos\theta - \left.\frac{2a^3}{r^3}E_0\cos\theta\right|_{r=a}$$

$$= -E_0\cos\theta - 2E_0\cos\theta = -3E_0\cos\theta = -\sigma/\epsilon_0$$

thus, we find,

$$\sigma(\theta) = 3\varepsilon_0 E_0 \cos\theta$$



so the + charge accumulates near the "north pole" and − charge does near the south pole as we might expect.