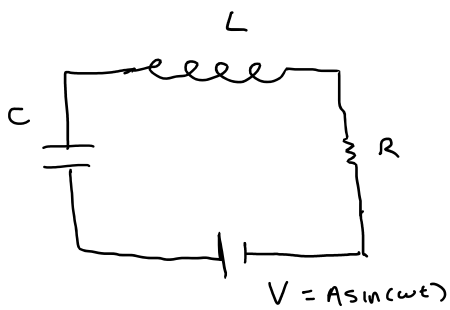
Sawford_Project1

Part 1: The Van der Pol Oscillator

Consider the RLC circuit shown below, where for some voltage *V,* inductance *L* and some capacitance *C*, and the resistor $R = I^2 - \alpha$ is non-Ohmic.



By using Kirchhoff's Voltage Laws, we can write the current through the circuit as the second order differential equation below, which can be used to describe the current oscillations through a vacuum tube.

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = V$$

$$L\frac{dI}{dt} + (I^2 - \alpha)I + \frac{1}{C}Q = 0$$

$$L\frac{d^2I}{d^2t} + 3\frac{dI}{dt}\left(I^2 - \frac{\alpha}{3}\right)I + \frac{1}{C}I = 0$$

We can then non-dimensionalize this to take the form of the Van der Pol Differential Equation, as shown below.

$$\frac{d^2x}{d^2t} + \mu(1 - x^2)\frac{dx}{dt} = 0$$

The rest of this project is as follows: Part 2 will describe the assumptions made for the system. Part 3 will discuss the fixed points and their stability for 4 different scenarios corresponding to different values of μ. Part 4 will be in the Jupyter Notebook attached and will contain the computational results.

Part 2: Assumptions and Limitations

We assume the resistor is non-Ohmic, meaning it doesn't follow Ohm's Law of $V = IR$. If the resistor were Ohmic, the non-dimensionalized second-order differential equation would be that of a simple harmonic oscillator, which we have studied in class. In this case, the resistor would completley remove energy when the current is too high. Instead, the non-Ohmic resistor only damps the energy of the circuit when the current is too high, so we must use a damping term μ.

We are also assuming there voltage is DC so there is no driving force from an additional voltage, and no sinusoidal factors. Also, as a step during the non-dimensionalization, we assume the inductor *L* has an inductance of *L* = *1*. This allows us to make the simplification from $V = L\frac{dI}{dt}$ to $V = \frac{dI}{dt}$.

We assume that Kirchhoff's Law holds, which implies the charge through the current must be conserved. We make this assumption so we can assume the sum of all the voltage changes through the circuit is equal to 0. We also assume the magnetic field that results from the changing current is negligible. If we did not make this simplification, we would need to account for how the magnetic field induces a current in addition to the current produced by the voltage source. Along the same lines, we are assuming none of the electrical components are interacting with each other

Part 3: The Analytical Approach

In this section, we will determine the fixed points of the Van der Pol differential equation and their stability.

$$f = \dot{x} = v \qquad g = \dot{v} = -\mu(x^2-1)v - x$$

$$\dot{x} = v = 0 \qquad \dot{v} = -\mu(x^2-1)(0) - x = 0$$

$$(x_*, v_*) = (0,0)$$

$$A = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial v} \end{bmatrix}\Bigg|_{(x_*,v_*)} = \begin{bmatrix} 0 & 1 \\ -2\mu v x - 1 & -\mu(x^2-1) \end{bmatrix}\Bigg|_{(x_*,v_*)} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$$

Find eigenvalues:

$$\begin{vmatrix} -\lambda & 1 \\ -1 & \mu-\lambda \end{vmatrix} = -\lambda(\mu-\lambda) + 1 = 0$$

$$\lambda^2 - \lambda\mu + 1 = 0$$

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2-4}}{2}$$

Underdamped $\mu = .05$:

$$\lambda_{1,2} = \frac{.05 \pm \sqrt{.0025 - 4}}{2} = .025 \pm .999687i$$

$Re(\lambda_{1,2}) = .025 > 0 \Rightarrow$ unstable

Critically Damped $\mu = 1$:

$$\lambda_{1,2} = \frac{.05 \pm \sqrt{1 - 4}}{2} = .025 \pm .866025i$$

$Re(\lambda_{1,2}) = .025 > 0 \Rightarrow$ unstable

Overdamped $\mu = 5$:

$$\lambda_{1,2} = \frac{.05 \pm \sqrt{25 - 4}}{2} = 2.31629, -2.26629$$

$\lambda_1 > 0$, $\lambda_2 < 0 \Rightarrow$ saddle point

References:

Derivation of the equation is based on Sara Sawford's (my) PHY 480 Final Paper on the Van der Pol Oscillator.

Functions used are directly taken from PHY 415, Day 14 Activity

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1 Project 1: Investigating the Van Der Pol Oscillator

1.1 Sara Sawford

1.1.1 PHY 415

1.2 Section 3: Computational Investigation

1.2.1 A. Plotting Phase Space Diagrams and Trajectories

Based on how the first order differential equations were created, the current I is proportional to x, and the voltage V is proportional to v.

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp
```

[21]:
```
# Inputs x,v, and damping factor mu
# Outputs first order differential equations at some point (x,v)
def VP_eqn(x, v, mu = 1.):
    xdot, vdot = [v, -mu*(x**2-1)*v-x]
    return xdot, vdot

# Inputs X, VX as meshgrids and damping factor mu
# Output xdot, vdot as arrays
def VP_phase(X, VX, mu):
    xdot, vdot = np.zeros(X.shape), np.zeros(VX.shape)
    Xlim, Ylim = X.shape
    for i in range(Xlim):
        for j in range(Ylim):
            xloc = X[i, j]
            yloc = VX[i, j]
            xdot[i,j], vdot[i,j] = VP_eqn(xloc, yloc,mu)
    return xdot, vdot
# Inputs time t, curr_vals = x,v, and dampign factor
# Outputs solutions of xdot, vdot as arrays
def VP_eqn_for_solve_ivp(t,curr_vals, mu=1):
    x, v = curr_vals
```

```
    xdot, vdot = VP_eqn(x,v,mu)
    return xdot,vdot
```

[75]:
```
# Inputs value of damping factor mu
# No return value, but plots phase space and trajectories of x and v
def cases(mu, x ,v):
    tmax = 50
    dt = 0.1
    tspan = (0,tmax)
    t = np.arange(0,tmax,dt)
    mu = mu
    initial_condition = [x,v]
    solved = solve_ivp(VP_eqn_for_solve_ivp,tspan,initial_condition,t_eval = t,␣
 ↪args = (mu,),method="LSODA")

    N = 40
    if mu == 5:
        x = np.linspace(-5., 5., N)
        v = np.linspace(-10., 10., N)
    else:
        x = np.linspace(-3., 3., N)
        v = np.linspace(-3., 3., N)
    X, V = np.meshgrid(x, v)

    xdot, vdot = VP_phase(X, V,mu)

    ax = plt.figure(figsize=(10,20))

    plt.subplot(3,2,1)
    ## Plot with Quiver
    Q = plt.streamplot(X, V, xdot, vdot, color='k', broken_streamlines=True)
    ## Plot trajectory and the starting location
    plt.plot(solved.y[0],solved.y[1],lw = 3,c = 'red')
    plt.title("Phase Plot of Van der Pol Oscillator")
    plt.xlabel("x")
    plt.ylabel("$\\dot{x}$")
    plt.grid()

    plt.subplot(3,2,2)
    plt.title("x vs t")
    plt.plot(t, solved.y[0])
    plt.xlabel("t")
    plt.ylabel("x")
    plt.grid()

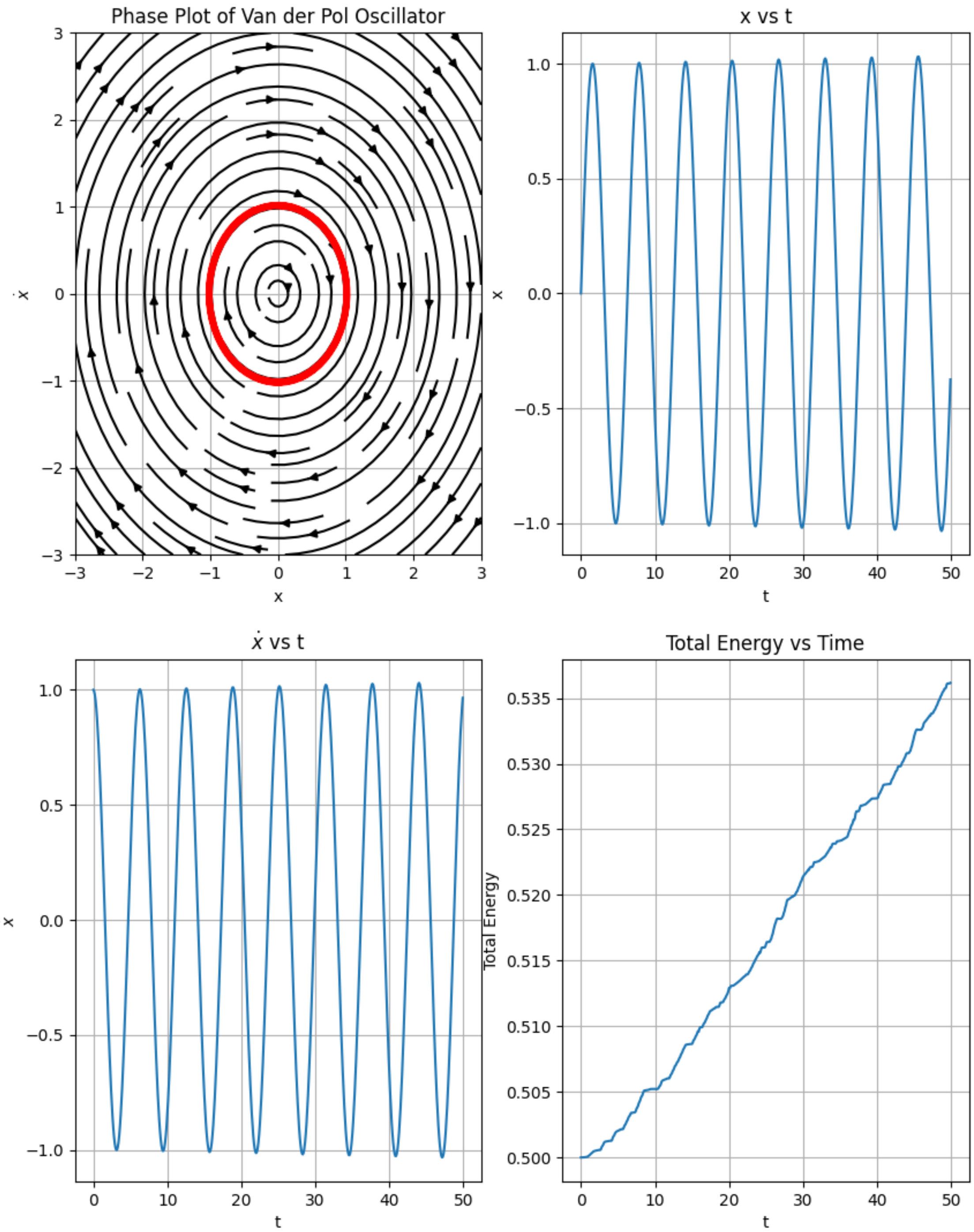
    plt.subplot(3,2,3)
    plt.title("$\\dot{x}$ vs t")
```

```
    plt.plot(t, solved.y[1])
    plt.xlabel("t")
    plt.ylabel("$\\dot{x}$")
    plt.grid()

    plt.subplot(3,2,4)
    # E_c = 1/2 * C * V^2
    # E_l = 1/2 * L * I^2
    plt.plot(t, .5*1*solved.y[1]**2+.5*1*solved.y[0]**2)
    plt.title("Total Energy vs Time")
    plt.xlabel("t")
    plt.ylabel("Total Energy")
    plt.grid()
```

1.2.2 No Damping Factor $\mu = 0$

[77]:
```
cases(0, 0 , 1)
```



Notice that since $\mu = 0$, the Van der Pol differential equation takes the form of a simple harmonic oscillator. The phase space diagram at the top has a closed orbit, which implies energy is conserved. This follows as we expect because when $\mu = 0$, the resistor becomes Ohmic, which implies the energy is changed linearly as the current changes across the resistor. We see this in the bottom right plot of Total Energy vs Time; even though there is a slight increase in totaly energy, it is only at 2.5%

change, which can be attributed to the numerical inaccuracy. We see in the second and third diagrams that the x and v plots (and so the current and voltage) are sinusoidal and vary with time, which makes sense since there is an inductor included in the RLC circuit.

If we started at (0,0), there would be no change in the current or voltage since (0,0) is a stable fixed point for a simple harmonic oscillator.

The family of solutions for this scenario are $Acos(\omega t) + Bsin(\omega t)$.

1.2.3 Underdamped Scenario $\mu = .05$

[78]:
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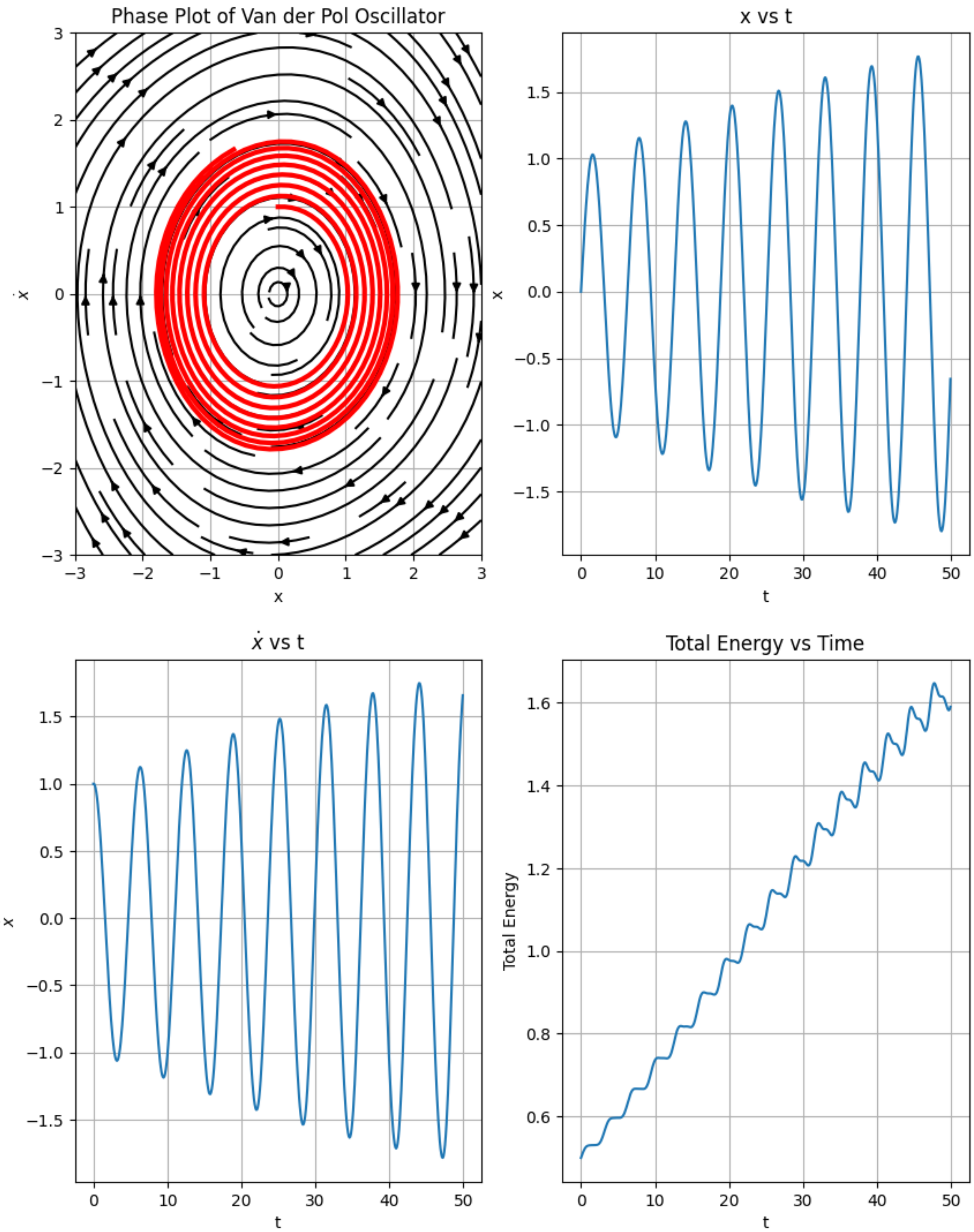
cases(.05, 0,1)
```



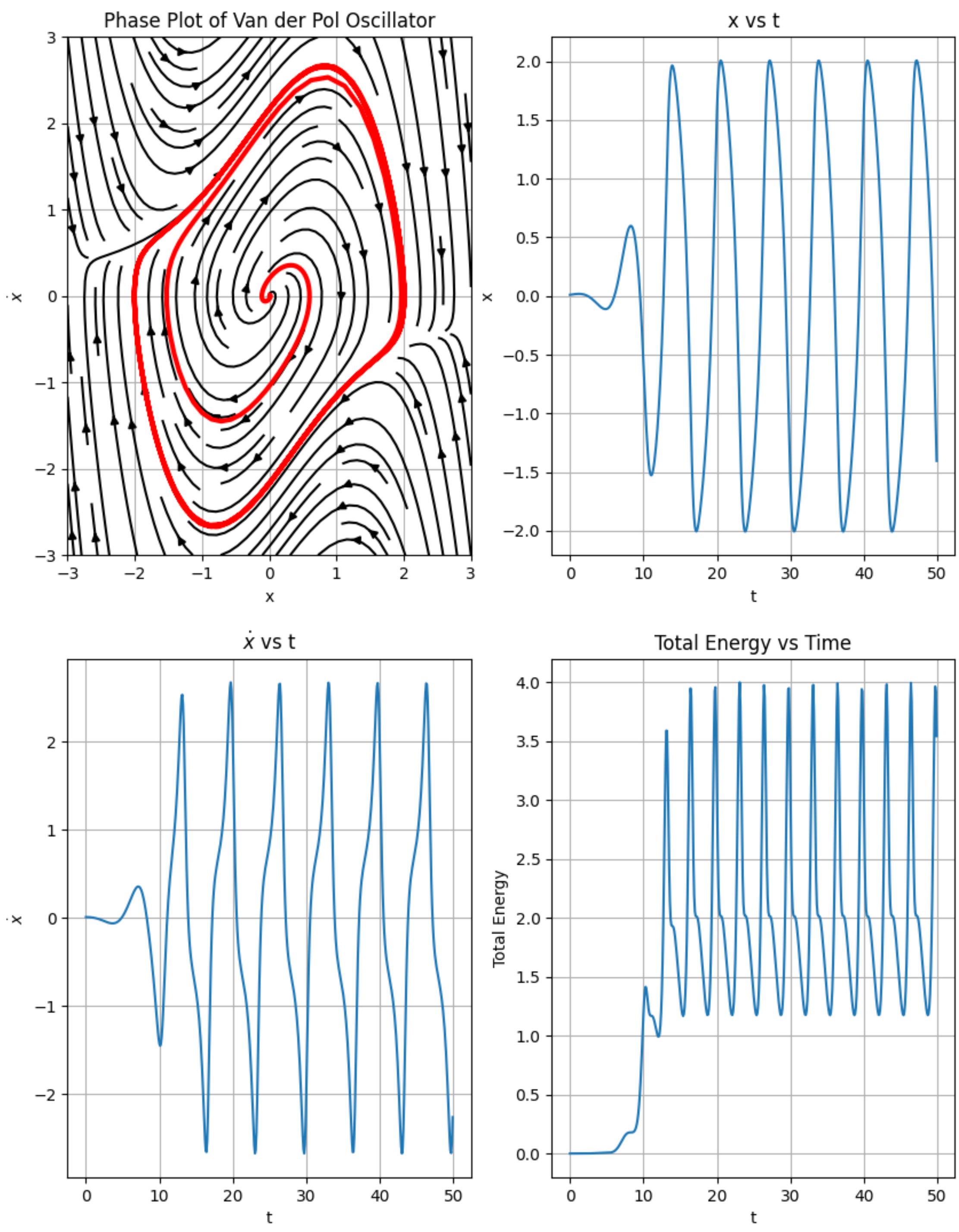
Based on the phase space diagram in the top plot, the trajectory is unstable since the trajectory spirals outward. This makes sense because the fixed point (0,0) calculated for $\mu = .05$ was unstable, which means the system approaches infinity as time increases. We see this in the next two plots, as the amplitude of each oscillation increases with each period. This indicates that energy is not conserved in this system, which is shown in the fourth plot of Energy vs time, which shows an 80%

change in energy after 6 cycles.

The family of solutions is approximately $x * sin(x)$, since the period doesn't change with each oscillation, but the amplitude increases.

1.2.4 Critically Damped $\mu = 1$

[80]:
```
cases(1, .01, .01)
```

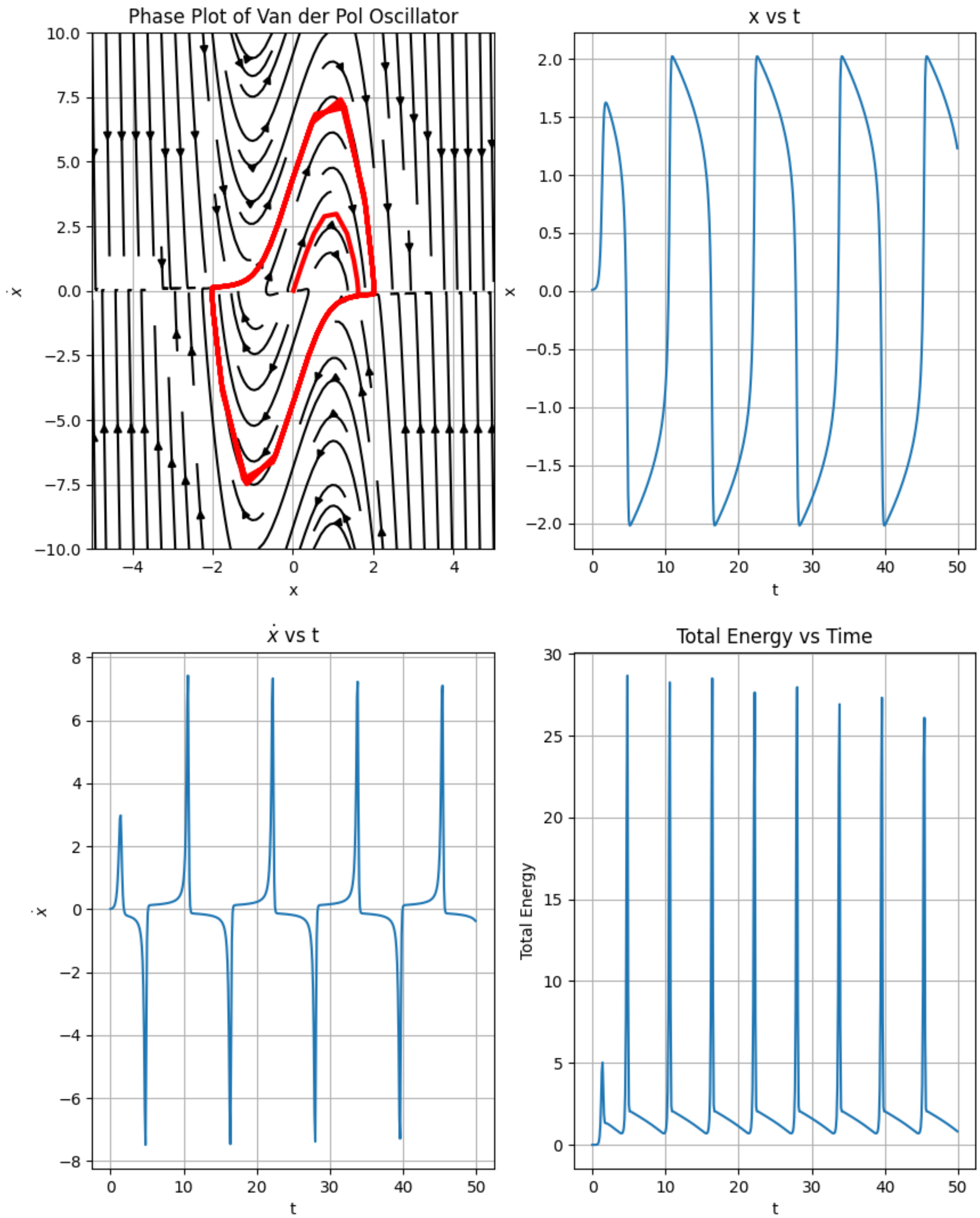


To see the phase space and trajectories the best, let our initial conditions be $x = .01$ and $v = .01$. From Part 3, we know the fixed point (0,0) is unstable for $\mu = 1$, which is shown in the phase space diagram. Once again, we see the phase space diagram spirals outward from the origin, so as time increases, the trajectory approaches infinity, which means energy is not conserved, as we can see in the bottom right plot. The energy increases from .5 J to 4 J within 10 seconds and then oscillates between 1.25 J and 4 J. We see the x and v vs time plots oscillate, and for the first three periods or so, the amplitude of oscillation changes, but after three periods, the amplitude is approximately the same.

The family of solutions (although not directly calculated) looks similar to the functions $Ae^x sin(x)$ since the first oscillation and the rest of the oscillations have a different period, which is similar to functions of the form $e^x sin(x)$.

1.2.5 Overdamped $\mu = 5$

[83]:
```
cases(5, .01, .01)
```



Again, we will use $x = .01$ and $v = .01$ to best see the change in trajectory and phase space diagram. The fixed point (0,0) is a saddle point based on the values of the eigenvalue of the Jacobian, which we can see in the phase space diagram; the phase space diagram points in opposite directions on opposite sides of 0. We still see that energy is not conserved, as follows on the fourth plot. The energy oscillates and the amplitude of the total energy changes with each period. The period of oscillation for the v vs t plot increased between this scenario and the critically damped scenario.

The family of solutions is probably similar to that of the critically damped scenario, although I am unsure of how the thin peaks would be modeled in any family of solutions.