Chapter 1 *Free Oscillations of Simple Systems*

1.1 Introduction

The world is full of things that move. Their motions can be broadly categorized into two classes, according to whether the thing that is moving stays near one place or travels from one place to another. Examples of the first class are an oscillating pendulum, a vibrating violin string, water sloshing back and forth in a cup, electrons vibrating (or whatever they do) in atoms, light bouncing back and forth between the mirrors of a laser. Parallel examples of traveling motion are a sliding hockey puck, a pulse traveling down a long stretched rope plucked at one end, ocean waves rolling toward the beach, the electron beam of a TV tube; a ray of light emitted at a star and detected at your eye. Sometimes the same phenomenon exhibits one or the other class of motion (i.e., standing still on the average, or traveling) depending on your point of view: the ocean waves travel toward the beach, but the water (and the duck sitting on the surface) goes up and down (and also forward and backward) without traveling. The displacement pulse travels down the rope, but the material of the rope vibrates without traveling.

We begin by studying things that stay in one vicinity and oscillate or vibrate about an average position. In Chaps. 1 and 2 we shall study many examples of the motion of a closed system that has been given an initial excitation (by some external disturbance) and is thereafter allowed to oscillate freely without further influence. Such oscillations are called *free* or *natural oscillations*. In Chap. 1 study of these simple systems having one or two moving parts will form the basis for our understanding of the free oscillations of systems with many moving parts in Chap. 2. There we shall find that the motion of a complicated system having many moving parts may always be regarded as compounded from simpler motions, called *modes*, all going on at once. No matter how complicated the system, we shall find that each one of its modes has properties very similar to those of a simple harmonic oscillator. Thus for motion of any system in a single one of its modes, we shall find that each moving part experiences the same return force per unit mass per unit displacement and that all moving parts oscillate with the same time dependence $\cos(\omega t + \varphi)$, i.e., with the same frequency ω and the same phase constant φ.

Each of the systems that we shall study is described by some physical quantity whose displacement from its equilibrium value varies with position in the system and time. In the mechanical examples (involving moving parts which are point masses subject to return forces), the physical quantity is the displacement of the mass at the point x,y,z from its equilibrium position. The displacement is described by a vector $\psi(x,y,x,t)$. Sometimes we call this vector function of x, y, z, t a *wave function.* (It is only a continuous function of x, y, and z when we can use the continuous approximation, i.e., when near neighbors have essentially the same motion.) In some of the electrical examples, the physical quantity may be the current in a coil or the charge on a capacitor. In others, it may be the electric field $\mathbf{E}(x,y,z,t)$ or the magnetic field $\mathbf{B}(x,y,z,t)$. In the latter cases, the waves are called electromagnetic waves.

1.2 Free Oscillations of Systems with One Degree of Freedom

We shall begin with things that stay in one vicinity, oscillating or vibrating about an average position. Such simple systems as a pendulum oscillating in a plane, a mass on a spring, and an LC circuit, whose configuration at any time can be completely specified by giving a single quantity, are said to have one degree of freedom—loosely speaking, one moving part (see Fig. 1.1). For example, the swinging pendulum can be described by the angle that the string makes with the vertical, and the LC circuit by the charge on the capacitor. (A pendulum free to swing in any direction, like a bob on a string, has not one but two degrees of freedom; it takes two coordinates to specify the position of the bob. The pendulum on a grandfather clock is constrained to swing in a plane, and thus has only one degree of freedom.)

For all these systems with one degree of freedom, we shall find that the displacement of the "moving part" from its equilibrium value has the same simple time dependence (called *harmonic oscillation*),

$$\psi(t) = A \cos(\omega t + \varphi). \tag{1}$$

For the oscillating mass, ψ may represent the displacement of the mass from its equilibrium position; for the oscillating LC circuit, it may represent the current in the inductor or the charge on the capacitor. More precisely, we shall find Eq. (1) gives the time dependence provided the moving part does not move too far from its equilibrium position. [For large-angle swings of a pendulum, Eq. (1) is a poor approximation to the motion; for large extensions of a real spring, the return force is not proportional to the extension, and the motion is not given by Eq. (1); a large enough charge on a capacitor will cause it to "break down" by sparking between the plates, and the charge will not satisfy Eq. (1).]

Nomenclature. We use the following nomenclature with Eq. (1): A is a positive constant called the *amplitude;* ω is the *angular frequency,* measured in radians per second; $\nu = \omega/2\pi$ is the *frequency,* measured in cycles per

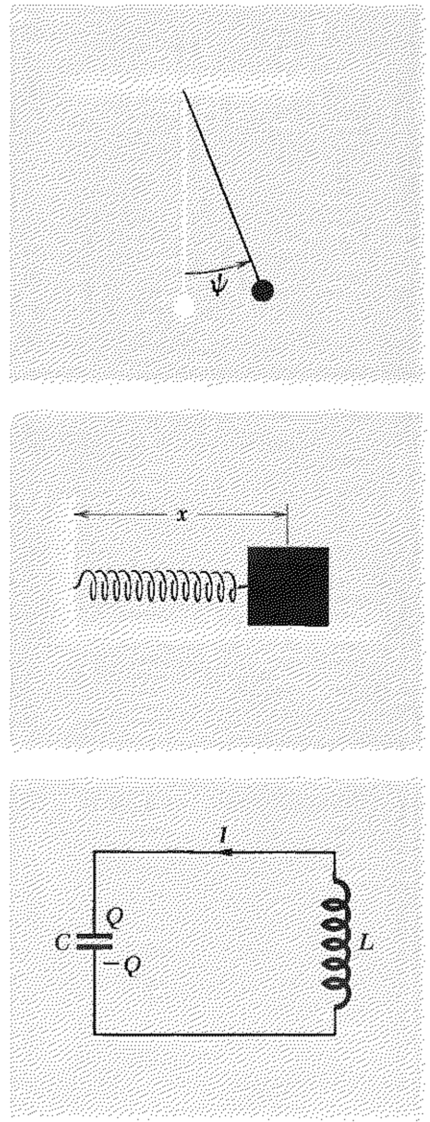

***Fig. 1.1** Systems with one degree of freedom. (The pendulum is constrained to swing in a plane.)*

second, or hertz (abbreviated cps, or Hz). The inverse of ν is called the *period* T, which is given in seconds per cycle:

$$T = \frac{1}{\nu}. \tag{2}$$

The *phase constant* φ corresponds to the choice of the zero of time. Often we are not particularly interested in the value of the phase constant. In these cases we can always "reset the clock," so that φ becomes zero, and then we write $\psi = A \cos \omega t$ or $\psi = A \sin \omega t$, instead of the more general Eq. (1).

Return force and inertia. The oscillatory behavior represented by Eq. (1) always results from the interplay of two intrinsic properties of the physical system which have opposite tendencies: *return force* and *inertia.* The "return force" tries to return ψ to zero by imparting a suitable "velocity" $d\psi/dt$ to the moving part. The greater ψ is, the stronger the return force. For the oscillating LC circuit, the return force is due to the repulsive force between the electrons, which makes the electrons prefer not to crowd onto one of the capacitor plates, but rather to distribute themselves equally on each plate, giving zero charge. The second property, "inertia," "opposes" any change in $d\psi/dt$. For the oscillating LC circuit, the inertia is due to the inductance L, which opposes any change in the current $d\psi/dt$ (letting ψ stand for the charge on the capacitor).

Oscillatory behavior. If we start with ψ positive and $d\psi/dt$ zero, the return force gives an acceleration which induces a negative velocity. By the time ψ returns to zero, the negative velocity is maximum. The return force is zero at $\psi = 0$, but the negative velocity now induces a negative displacement. Then the return force becomes positive, but it must now overcome the inertia of the negative velocity. Finally, the velocity $d\psi/dt$ is zero, but by that time the displacement is large and negative, and the process reverses. This cycle goes on and on: the return force tries to restore ψ to zero; in so doing, it induces a velocity; the inertia preserves the velocity and causes ψ to "overshoot." The system oscillates.

Physical meaning of ω^2. The angular frequency of oscillation ω is related to the physical properties of the system in every case (as we shall show) by the relation

$$\boxed{\omega^2 = \text{return force per unit displacement per unit mass.}} \tag{3}$$

Sometimes, as in the case of the electrical examples (LC circuit), the "inertial mass" may not actually be mass.

Damped oscillations. If left undisturbed, an oscillating system will continue to oscillate forever in accordance with Eq. (1). However, in any real physical situation, there are "frictional," or "resistive," processes which "damp" the motion. Thus a more realistic description of an oscillating system is given by a "damped oscillation." If the system is "excited" into oscillation at $t = 0$ (by giving it a bump or closing a switch or something), we find (see Vol. I, Chap. 7, page 209)

$$\psi(t) = Ae^{-t/2\tau}\cos(\omega t + \varphi), \tag{4}$$

for $t \geqq 0$, with the understanding that ψ is zero for $t < 0$. For simplicity we shall use Eq. (1) instead of the more realistic Eq. (4) in the examples that follow. We are thus neglecting friction (or resistance in the case of the LC circuit) by taking the decay time τ to be infinite.

Example 1: Pendulum

A simple pendulum consists of a massless string or rod of length l attached at the top to a rigid support and at the bottom to a "point" bob of mass M (see Fig. 1.2). Let ψ denote the angle (in radians) that the string makes with the vertical. (The pendulum swings in a plane; its configuration is given by ψ alone.) The displacement of the bob, as measured along the perimeter of the circular arc of its path, is $l\psi$. The corresponding instantaneous tangential velocity is $l\,d\psi/dt$. The corresponding tangential acceleration is $l\,d^2\psi/dt^2$. The return force is the tangential component of force. The string does not contribute to this force component. The weight Mg contributes the tangential component $-Mg\sin\psi$. Thus Newton's second law (mass times acceleration equals force) gives

$$\frac{Ml\,d^2\psi}{dt^2} = -Mg\sin\psi(t). \tag{5}$$

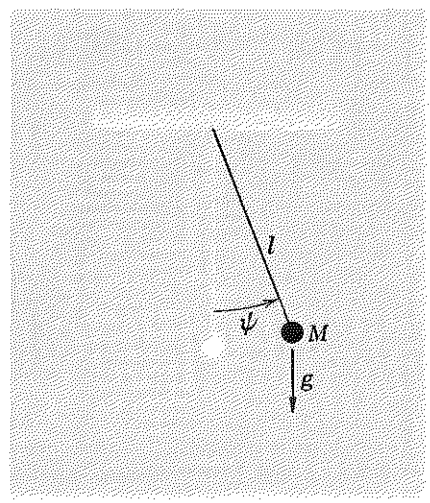

Fig. 1.2 Simple pendulum.

We now use the Taylor's series expansion [Appendix, Eq. (4)]

$$\sin\psi = \psi - \frac{\psi^3}{3!} + \frac{\psi^5}{5!} - \cdots, \tag{6}$$

where the ellipsis ($\cdots$) denotes the rest of the infinite series. We see that for sufficiently small ψ (in radians, remember), we can neglect all terms in Eq. (6) except the first one, ψ. You may ask, "How small is 'sufficiently small'?" That question has no universal answer—it depends on how accurately you can determine the function $\psi(t)$ in the experiment you have in mind (this is physics, remember—nothing is perfectly measurable) and on how much you care. For example, for $\psi = 0.10$ rad (5.7 deg), $\sin\psi$ is 0.0998; in some problems "$0.0998 = 0.1000$" is a poor approximation. For $\psi = 1.0$ rad (57.3 deg), $\sin\psi$ is 0.841; in some problems "$0.8 = 1.0$" is an adequate approximation.

If we retain only the first term in Eq. (6), then Eq. (5) takes on the form

$$\frac{d^2\psi}{dt^2} = -\omega^2\psi, \tag{7}$$

where

$$\omega^2 = \frac{g}{l}. \tag{8}$$

The general solution of Eq. (7) is the harmonic oscillation given by

$$\psi(t) = A\cos(\omega t + \varphi).$$

Note that the angular frequency of oscillation, given by Eq. (8), can be written

$$\omega^2 = \text{return force per unit displacement per unit mass:}$$

$$\omega^2 = \frac{Mg\psi}{(l\psi)M} = \frac{g}{l},$$

using the approximation that $\sin\psi$ equals ψ.

The two constants A and φ are determined by the initial conditions, i.e., by the displacement and velocity at time $t = 0$. (Since ψ is an angular displacement, the corresponding "velocity" is the angular velocity $d\psi/dt$.) Thus we have

$$\psi(t) = A\cos(\omega t + \varphi),$$

$$\dot\psi(t) \equiv \frac{d\psi(t)}{dt} = -\omega A\sin(\omega t + \varphi),$$

so that

$$\psi(0) = A\cos\varphi,$$

$$\dot\psi(0) = -\omega A\sin\varphi.$$

These two equations may be solved for the positive constant A and for $\sin\varphi$ and $\cos\varphi$ (which determine φ).

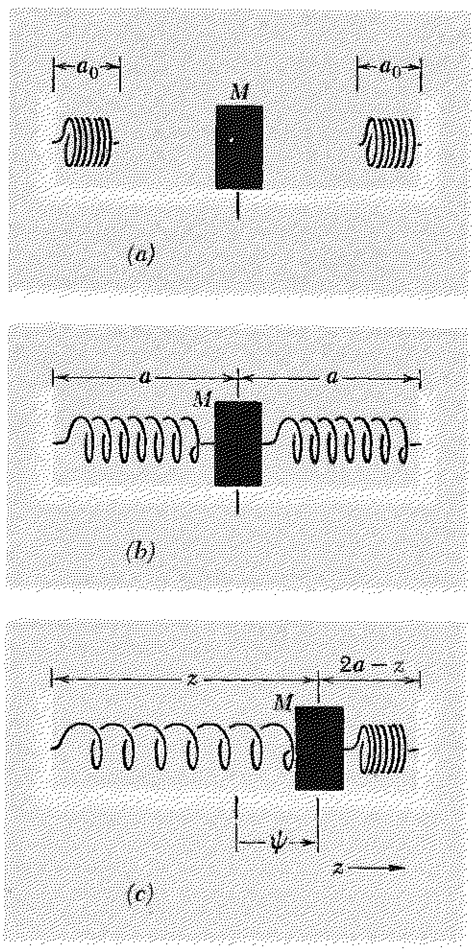

Fig. 1.3 Longitudinal oscillations. (a) Springs relaxed and unattached. (b) Springs attached, M at equilibrium position. (c) General configuration.

Example 2: Mass and springs—longitudinal oscillations

Mass M slides on a frictionless surface. It is connected to rigid walls by means of two identical springs, each of which has zero mass, spring constant K, and relaxed length a_0. At the equilibrium position, each spring is stretched to length a, and thus each spring has tension $K(a - a_0)$ at equilibrium (see Fig. 1.3*a* and *b*). Let z be the distance of M from the left-hand wall. Then its distance from the right-hand wall is $2a - z$ (see Fig. 1.3*c*). The left-hand spring exerts a force $K(z - a_0)$ in the $-z$ direction. The right-hand spring exerts a force $K(2a - z - a_0)$ in the $+z$ direction. The total force F_z in the $+z$ direction is the superposition (sum) of these two forces:

$$F_z = -K(z - a_0) + K(2a - z - a_0)$$
$$= -2K(z - a).$$

Newton's second law then gives

$$\frac{M\,d^2z}{dt^2} = F_z = -2K(z - a). \tag{9}$$

The displacement from equilibrium is $z - a$. We designate this by $\psi(t)$:

$$\psi(t) \equiv z(t) - a.$$

then

$$\frac{d^2\psi}{dt^2} = \frac{d^2z}{dt^2}.$$

Now we can write Eq. (9) in the form

$$\frac{d^2\psi}{dt^2} = -\omega^2\psi, \tag{10}$$

with

$$\omega^2 = \frac{2K}{M}. \tag{11}$$

The general solution of Eq. (10) is again the harmonic oscillation $\psi = A\cos(\omega t + \varphi)$. Note that Eq. (11) has the form ω^2 = force per unit displacement per unit mass, since the return force is $2K\psi$ for a displacement ψ.

Example 3: Mass and springs—transverse oscillations

The system is shown in Fig. 1.4. Mass M is suspended between rigid supports by means of two identical springs. The springs each have zero mass, spring constant K, and unstretched length a_0. They each have length a at the equilibrium position of M. We neglect the effect of gravity. (Gravity does not produce any return force in this problem. It does cause the system to "sag," but that does not affect the results in the order of approximation that we are interested in.) Mass M now has three degrees of freedom: It can move in the z direction (along the axis of the springs) to give "longitudinal" oscillation. That is the motion we considered above, and we need not repeat those considerations. It can also move in the x direction or in the y direction to give "transverse" oscillations. For simplicity, let us consider only motion along x. We may imagine that there is some frictionless constraint that allows complete freedom of motion in the transverse x direction but prevents motion along either y or z. (For example, we could

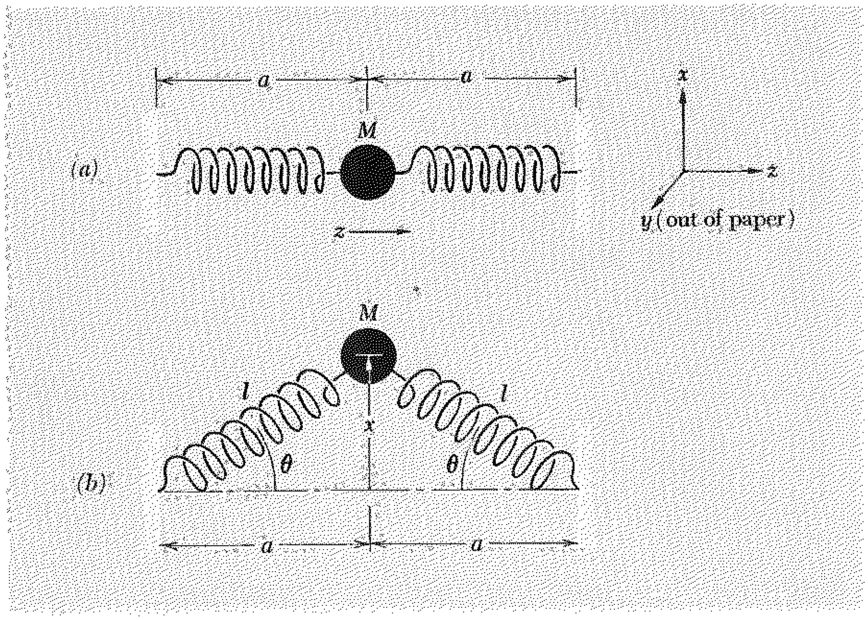

Fig. 1.4 Transverse oscillations. (a) Equilibrium configuration. (b) General configuration (for motion along x).

drill a hole through M and arrange a frictionless rod passing through the hole, rigidly attached to the walls, and oriented along x. However, you can easily convince yourself that such a constraint is unnecessary. From the symmetry of Fig. 1.4, you can see that if at a given time the system is oscillating along x, there is no tendency for it to acquire any motion along y or z. The same circumstance holds true for each of the other two degrees of freedom: no unbalanced force along x or y is developed due to oscillation along z, nor along x or z due to oscillation along y.)

At equilibrium (Fig. 1.4*a*), each of the springs has length a and exerts a tension T_0, given by

$$T_0 = K(a - a_0). \tag{12}$$

In the general configuration (Fig. 1.4*b*), each spring has length l and tension

$$T = K(l - a_0). \tag{13}$$

This tension is exerted along the axis of the spring. Taking the x component of this force, we see that each spring contributes a return force $T \sin\theta$ in the $-x$ direction. Using Newton's second law and the fact that $\sin\theta$ is x/l, we find

$$\begin{aligned} M\frac{d^2x}{dt^2} &= F_x = -2T\sin\theta \\ &= -2K(l - a_0)\frac{x}{l} = -2Kx\left(1 - \frac{a_0}{l}\right). \end{aligned} \tag{14}$$

Equation (14) is exact, under our assumptions (including the assumption, expressed by Eq. (13), that the spring is a "linear" or "Hooke's law" spring). Notice that the spring length l which appears on the right side of Eq. (14) is a function of x. Therefore Eq. (14) is not exactly of the form that gives rise to harmonic oscillations, because the return force on M is not exactly linearly proportional to the displacement from equilibrium, x.

Slinky approximation. There are two interesting ways in which we can obtain an approximate equation with a linear restoring force. The first way we shall call the *slinky approximation*, in which we neglect a_0/a compared to unity. Hence, since l is always greater than a, we neglect a_0/l in Eq. (14). [A slinky is a helical spring with relaxed length a_0 about 3 inches. It can be stretched to a length a of about 15 feet without exceeding its elastic limit. That would give $a_0/a < 1/60$ in Eq. (14).] Using this approximation, we can write Eq. (14) in the form

$$\frac{d^2x}{dt^2} = -\omega^2 x, \tag{15}$$

with

$$\omega^2 = \frac{2K}{M} = \frac{2T_0}{Ma} \qquad (\text{for } a_0 = 0). \tag{16}$$

This has the solution $x = A\cos(\omega t + \varphi)$, i.e., harmonic oscillation. Notice that there is no restriction on the amplitude A. We can have "large" oscillations and still have perfect linearity of the return force. Notice also that the frequency for transverse oscillations, as given by Eq. (16), is the same as that for longitudinal oscillations, as given by Eq. (11). That is not true in general. It holds only in the slinky approximation, where we effectively take $a_0 = 0$.

Small-oscillations approximation. If a_0 cannot be neglected with respect to a (as is the case, for example, with a rubber rope under the conditions ordinarily met in lecture demonstrations), the slinky approximation does not apply. Then F_x in Eq. (14) is not linear in x. However, we shall show that if the displacements x are small compared with the length a, then l differs from a only by a quantity of order $a(x/a)^2$. In the *small-oscillations approximation*, we neglect the terms in F_x which are nonlinear in x/a. Let us now do the algebra: We want to express l in Eq. (14) as $l = a +$ something, where "something" vanishes when $x = 0$. Since l is larger than a, whether x is positive or negative, "something" must be an even function of x. In fact we have from Fig. 1.4

$$\begin{aligned} l^2 &= a^2 + x^2 \\ &= a^2(1+\epsilon), \qquad \epsilon \equiv \frac{x^2}{a^2}. \end{aligned}$$

Thus

$$\frac{1}{l} = \frac{1}{a}(1 + \epsilon)^{-(1/2)}$$
$$= \frac{1}{a}\left[1 - \left(\frac{1}{2}\epsilon\right) + \left(\frac{3}{8}\epsilon^2\right) - \cdots\right], \tag{17}$$

where we have used the Taylor's series expansion [Appendix Eq. (20)] for $(1 + x)^n$ with $n = -\frac{1}{2}$ and $x = \epsilon$. Next we make the small-oscillations approximation. We assume we have $\epsilon \ll 1$ and discard the higher-order terms in the infinite series of Eq. (17). (Eventually we shall drop everything except the first term, $1/a$.) Thus we have

$$\frac{1}{l} \approx \frac{1}{a}\left[1 - \left(\frac{1}{2}\epsilon\right)\right]$$
$$= \frac{1}{a}\left[1 - \left(\frac{1}{2}\frac{x^2}{a^2}\right)\right]. \tag{18}$$

Inserting Eq. (18) into Eq. (14), we find

$$\frac{d^2x}{dt^2} = -\frac{2Kx}{M}\left(1 - \frac{a_0}{l}\right)$$
$$= -\frac{2Kx}{M}\left\{1 - \frac{a_0}{a}\left[1 - \left(\frac{1}{2}\frac{x^2}{a^2}\right) + \cdots\right]\right\}$$
$$= -\frac{2K}{Ma}(a - a_0)x + \frac{K}{M}a_0\left(\frac{x}{a}\right)^3 + \cdots. \tag{19}$$

Discarding the cubic and higher-order terms, we obtain

$$\frac{d^2x}{dt^2} \approx -\frac{2K}{Ma}(a - a_0)x = -\frac{2T_0x}{Ma}. \tag{20}$$

[In the second equality of Eq. (20), we used T_0 as given by Eq. (12).] Equation (20) is of the form

$$\frac{d^2x}{dt^2} = -\omega^2 x,$$

with

$$\omega^2 = \frac{2T_0}{Ma}. \tag{21}$$

Therefore $x(t)$ is given by the harmonic oscillation

$$x(t) = A\cos(\omega t + \varphi).$$

Notice that ω^2 given by Eq. (21) is the return force per unit displacement per unit mass: for small oscillations, the return force is the tension T_0 times $\sin\theta$, which is x/a, times two (two springs). The displacement is x; the mass is M. Thus the return force per unit displacement per unit mass is $2T_0(x/a)/xM$.

Notice that the frequency for transverse oscillations is given by $\omega^2 = 2T_0/Ma$ for both the case of the slinky approximation ($a_0 = 0$) and the small-oscillations approximation ($x/a \ll 1$), as we see by comparing Eqs. (16) and (21). In the slinky approximation, the longitudinal oscillation also has this same frequency, as we see from Eqs. (11) and (16). If the slinky approximation does not hold (i.e., if a_0/a cannot be neglected), then the longitudinal oscillations and (small) transverse oscillations do not have the same frequency, as we see from Eqs. (11), (12), and (21). In this case,

$$(\omega^2)_{\text{long}} = \frac{2Ka}{Ma}, \tag{22}$$

$$(\omega^2)_{\text{tr}} = \frac{2T_0}{Ma}, \qquad T_0 = K(a - a_0). \tag{23}$$

Thus for small oscillations of a rubber rope (where a_0/a cannot be neglected), the longitudinal oscillations are more rapid than the transverse oscillations:

$$\frac{\omega_{\text{long}}}{\omega_{\text{tr}}} = \frac{1}{\left[1 - \dfrac{a_0}{a}\right]^{1/2}}.$$

Example 4: *LC* circuit

(For a more complete discussion of LC circuits, see Vol. 2, Chap. 8.) Consider the series LC circuit of Fig. 1.5. The charge displaced from the bottom to the top plate of the left-hand capacitor is Q_1. That displaced from bottom to top of the right-hand capacitor is Q_2. The electromotive force (emf) across the inductance is equal to the "back emf," $L\,dI/dt$. Charge Q_1 provides an electromotive force equal to $C^{-1}Q_1$, such that positive Q_1 drives current in the direction of the arrow in Fig. 1.5. Thus positive Q_1 gives positive $L\,dI/dt$. Similarly, from Fig. 1.5, positive Q_2 gives negative $L\,dI/dt$. Thus we have

$$L\frac{dI}{dt} = C^{-1}Q_1 - C^{-1}Q_2. \tag{24}$$

At equilibrium there is no charge on either capacitor. The charge Q_2 is built up by the current I at the expense of the charge Q_1. Thus, using charge conservation and the sign conventions of Fig. 1.5, we have

$$Q_1 = -Q_2, \tag{25}$$

$$\frac{dQ_2}{dt} = I. \tag{26}$$

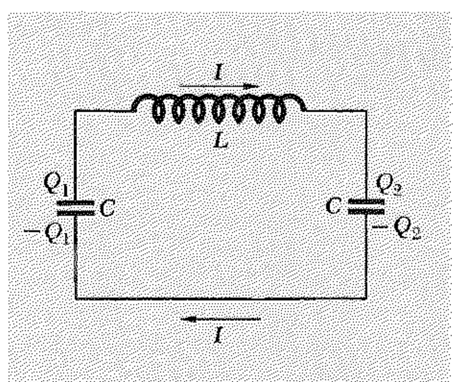

***Fig. 1.5** Series LC circuit. The sign conventions for Q and I are indicated. Q_1 (or Q_2) is positive if the upper plate is positive with respect to the lower plate; I is positive if positive charge flows in the direction of the arrows.*

Because of Eqs. (25) and (26), there is only one degree of freedom. We can describe the instantaneous configuration of the system by giving Q_1, or Q_2, or I. The current I will be most convenient in our later work (when we go to systems having more than one degree of freedom), and we shall use it here. We first use Eq. (25) to eliminate Q_1 from Eq. (24); then we differentiate with respect to t and use Eq. (26) to eliminate Q_2:

$$L\frac{dI}{dt} = C^{-1}Q_1 - C^{-1}Q_2 = -2C^{-1}Q_2;$$

$$L\frac{d^2I}{dt^2} = -2C^{-1}\frac{dQ_2}{dt} = -2C^{-1}I.$$

Thus the current $I(t)$ obeys the equation

$$\frac{d^2I}{dt^2} = -\omega^2 I,$$

with

$$\omega^2 = \frac{2C^{-1}}{L}, \tag{27}$$

and $I(t)$ undergoes harmonic oscillation:

$$I(t) = A\cos(\omega t + \varphi).$$

We can think of Eq. (27) as an illustration of the fact that ω^2 is always the "return force" per unit "displacement" per unit "inertia." We can take the "return force" to be the electromotive force $2C^{-1}Q$, where Q is the "charge displacement" Q_2. We then take the self-inductance L to be the "charge inertia." Then the return force per unit displacement per unit inertia is $(2C^{-1}Q)/QL$.

You may have noticed a mathematical parallelism between Examples 2, 3, and 4. We purposely gave these examples the same spatial symmetry ("inertia" in the center, "driving forces" located symmetrically on either side) so as to produce the parallelism. Such parallelisms are often useful as mnemonic devices.

1.3 *Linearity and the Superposition Principle*

In Sec. 1.2 we solved for the oscillations of the pendulum and of the mass and springs only for the cases where we could assume the return force to be proportional to $-\psi$, with (for example) no dependence on ψ^2, ψ^3, etc. A differential equation that contains no higher than the first power of ψ, of $d\psi/dt$, of $d^2\psi/dt^2$, etc., is said to be *linear* in ψ and its time derivatives. If, in addition, no terms independent of ψ occur, the equation is said to be *homogeneous* as well. If any higher powers of ψ or its derivatives occur in the equation, the equation is said to be *nonlinear*. For example, Eq. (5) is nonlinear, as we can see from the expansion of sin ψ given by Eq. (6). Only when we neglect the higher powers of ψ do we obtain a linear equation.

Nonlinear equations are generally difficult to solve. (The nonlinear pendulum equation is solved exactly in Volume I, pp. 225 ff.) Fortunately, there are many interesting physical situations for which linear equations give a very good approximation. We shall deal almost entirely with linear equations.

Linear homogeneous equations. Linear homogeneous differential equations have the following very interesting and important property: *The sum of any two solutions is itself a solution.* Nonlinear equations do not have that property. The sum of two solutions of a nonlinear equation is not itself a solution of the equation.

We shall prove these statements for both cases (linear and nonlinear) at once. Suppose that we have found the differential equation of motion of a system with one degree of freedom to be of the form

$$\frac{d^2\psi(t)}{dt^2} = -C\psi + \alpha\psi^2 + \beta\psi^3 + \gamma\psi^4 + \cdots, \tag{28}$$

as we found, for example, for the pendulum [Eqs. (5) and (6)] or for the transverse oscillations of a mass suspended by springs [Eq. (19)]. If the constants α, β, γ, etc. are all zero or can be taken to be zero as a sufficiently good approximation, then Eq. (28) is linear and homogeneous. Otherwise, it is nonlinear. Now suppose that $\psi_1(t)$ is a solution of Eq. (28) and that $\psi_2(t)$ is a different solution. For example, ψ_1 may be the solution corresponding to a particular initial displacement and initial velocity of a pendulum bob, and ψ_2 may correspond to different initial displacement and velocity. By hypothesis ψ_1 and ψ_2 each satisfy Eq. (28). Thus we have

$$\frac{d^2\psi_1}{dt^2} = -C\psi_1 + \alpha\psi_1{}^2 + \beta\psi_1{}^3 + \gamma\psi_1{}^4 + \cdots, \tag{29}$$

and

$$\frac{d^2\psi_2}{dt^2} = -C\psi_2 + \alpha\psi_2{}^2 + \beta\psi_2{}^3 + \gamma\psi_2{}^4 + \cdots. \tag{30}$$

The question of interest to us is whether or not the *superposition* of ψ_1 and ψ_2, defined as the sum $\psi(t) = \psi_1(t) + \psi_2(t)$, satisfies the same equation of motion, Eq. (28). Do we have

$$\frac{d^2(\psi_1 + \psi_2)}{dt^2} = -C(\psi_1 + \psi_2) + \alpha(\psi_1 + \psi_2)^2 + \beta(\psi_1 + \psi_2)^3 + \cdots ? \tag{31}$$

The question (31) has the answer "yes" if and only if the constants α, β, etc. are zero. That is easily shown as follows. Add Eqs. (29) and (30).

The sum gives Eq. (31) if and only if all the following conditions are satisfied:

$$\frac{d^2\psi_1}{dt^2} + \frac{d^2\psi_2}{dt^2} = \frac{d^2(\psi_1 + \psi_2)}{dt^2}, \tag{32}$$

$$-C\psi_1 - C\psi_2 = -C(\psi_1 + \psi_2), \tag{33}$$

$$\alpha\psi_1^2 + \alpha\psi_2^2 = \alpha(\psi_1 + \psi_2)^2, \tag{34}$$

$$\beta\psi_1^3 + \beta\psi_2^3 = \beta(\psi_1 + \psi_2)^3, \text{ etc.} \tag{35}$$

Equations (32) and (33) are both true. Equations (34) and (35) are not true unless α and β are zero. Thus we see that the superposition of two solutions is itself a solution if and only if the equation is linear.

The property that a superposition of solutions is itself a solution is unique to homogeneous linear equations. Oscillations that obey such equations are said to obey the *superposition principle*. We shall not study any other kind.

Superposition of initial conditions. As an example of the applications of the concept of superposition, consider the motion of a simple pendulum under small oscillations. Suppose that one has found a solution ψ_1 corresponding to a certain set of initial conditions (displacement and velocity) and another solution ψ_2 corresponding to a different set of initial conditions. Now suppose we prescribe a third set of initial conditions as follows: We *superpose the initial conditions* corresponding to ψ_1 and ψ_2. That means that we give the bob an initial displacement that is the algebraic sum of the initial displacement corresponding to the motion $\psi_1(t)$ and that corresponding to $\psi_2(t)$, and we give the bob an initial velocity that is the algebraic sum of the two initial velocities corresponding to ψ_1 and ψ_2. Then there is no need to do any more work to find the new motion, described by $\psi_3(t)$. The solution ψ_3 is just the superposition $\psi_1 + \psi_2$. We let you finish the proof. This result holds *only* if the pendulum oscillations are sufficiently small so that we can neglect the nonlinear terms in the return force.

Linear inhomogeneous equations. Linear *in*homogeneous equations (i.e., equations containing terms independent of ψ) also give rise to a superposition principle, though of a slightly different sort. There are many physical situations analogous to a driven harmonic oscillator, which satisfies the equation

$$\frac{M\, d^2\psi(t)}{dt^2} = -C\psi(t) + F(t), \tag{36}$$

where $F(t)$ is an "external" driving force that is independent of $\psi(t)$. The corresponding superposition principle is as follows: Suppose a driving force $F_1(t)$ produces an oscillation $\psi_1(t)$ (when F_1 is the only driving force), and suppose another driving force $F_2(t)$ produces an oscillation $\psi_2(t)$ [when $F_2(t)$

is present by itself]. Then, if both driving forces are present simultaneously [so that the total driving force is the superposition $F_1(t) + F_2(t)$], the corresponding oscillation [i.e., corresponding solution of Eq. (36)] is given by the superposition $\psi(t) = \psi_1(t) + \psi_2(t)$. We leave it to you to show that this is true for the linear inhomogeneous Eq. (36) and not true for an equation nonlinear in $\psi(t)$. (See Prob. 1.16.)

The systems we dealt with in Sec. 1.2 and our illustrations of the superposition principle in this section have all been systems of one degree of freedom. However, the superposition principle is applicable to systems of any number of degrees of freedom (when the equations are linear), and we shall be using it very often, usually without mentioning its name.

Example 5: Spherical pendulum

To illustrate the application of the superposition principle when we have two degrees of freedom, we consider the motion of a pendulum consisting of a bob of mass M on a string of length l. The pendulum is free to swing in any direction and is called a *spherical pendulum.* At equilibrium the string is vertical, along z, and the bob is at $x = y = 0$. For displacements x and y that are sufficiently small, you can easily show that $x(t)$ and $y(t)$ satisfy the differential equations

$$M\frac{d^2x}{dt^2} = -\frac{Mg}{l}x \tag{37}$$

$$M\frac{d^2y}{dt^2} = -\frac{Mg}{l}y. \tag{38}$$

These two equations are "uncoupled," by which we mean that the x component of force depends only on x, not on y, and vice versa. Thus Eq. (37) does not contain y, and similarly Eq. (38) does not contain x. Equations (37) and (38) can be solved independently to give

$$x(t) = A_1 \cos(\omega t + \varphi_1) \tag{39}$$

$$y(t) = A_2 \cos(\omega t + \varphi_2), \tag{40}$$

with

$$\omega^2 = \frac{g}{l},$$

where the constants A_1, A_2, φ_1, and φ_2 are determined by the initial conditions of displacement and velocity in the x and y directions. The complete motion can now be thought of as a *superposition* of the motion $\hat{\mathbf{x}}x(t)$ and the motion $\hat{\mathbf{y}}y(t)$, where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors. The power of the superposition principle lies in the fact that we can solve for the x and y motions separately and then merely superpose the two motions to get the complete motion involving both degrees of freedom.