Method of Relaxation in 1D

We are trying to solve Laplace's Equation

$$\nabla^2 V = 0$$

In some (many?) cases, it makes sense to solve it numerically because the solutions are not necessarily analytic functions. However, we can often exploit general property #3:

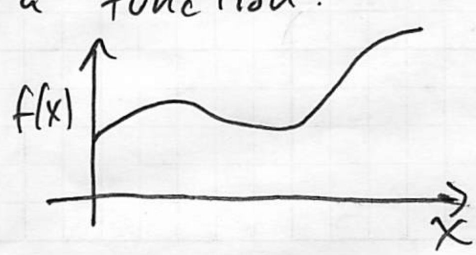
$$V(\vec{r}) = \frac{1}{4\pi R^2} \oiint V\,dA$$ V at a point is the average of it's neighbors.

We will approach this problem in 1D first, which arguably is boring b/c its always linear, because it will help us gain intuition about the algorithm and avoid common pitfalls.

1D Laplace Equation

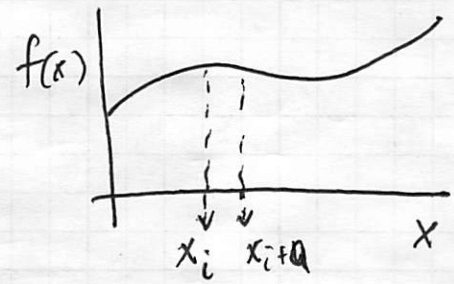
$V(x) = ?$ when $\frac{d^2V}{dx^2} = 0$

to develop a computational description of this problem we need to develop the concept of a numerical derivative of a function.



$\frac{df}{dx} = ?$

Consider some function $f(x)$,



Let's cut up the space into equally-sized steps of size a.

We can approximate the slope (derivative) of that function using a line,

$$\frac{df}{dx} \simeq \frac{f(x_i+a) - f(x_i)}{a}$$

* this would be the approx $\frac{df}{dx}$ @ $x_i + \frac{a}{2}$

So,

$$f'(x_i + \tfrac{a}{2}) \simeq \frac{f(x_i+a) - f(x_i)}{a} \quad \text{and}$$

$$f'(x_i - \tfrac{a}{2}) \simeq \frac{f(x_i) - f(x_i - a)}{a}$$

What about the second derivative, $f''(x_i)$?

given approximate values for f', we can further approximate f'' using the same ideas,

$$f''(x_i) \simeq \frac{f'(x_i + \frac{a}{2}) - f'(x_i - \frac{a}{2})}{a}$$

* This would be $\frac{d^2f}{dx^2}$ @ x_i b/c $\pm\frac{a}{2}$ are used as the bounds.

Can we write that in terms of $f(x)$ (as that's what we have)?

$$f''(x_i) \simeq \frac{f'(x_i+\frac{a}{2}) - f'(x_i-\frac{a}{2})}{a}$$

with
$$f'(x_i+\tfrac{a}{2}) \simeq \frac{f(x_i+a)-f(x_i)}{a}$$
$$f'(x_i-\tfrac{a}{2}) \simeq \frac{f(x_i)-f(x_i-a)}{a}$$

$$f''(x_i) \simeq \frac{[f(x_i+a)-f(x_i)]/a - [f(x_i)-f(x_i-a)]/a}{a}$$

$$= \frac{f(x_i+a) - 2f(x_i) + f(x_i-a)}{a^2}$$

What we have derived is a description of the numerical second derivative at a point x_i for the function f.

<u>Relating this to Laplace's Equation</u>

$$\nabla^2 V = 0 \Rightarrow \text{in 1D} \Rightarrow \frac{d^2V}{dx^2} = 0$$

$$\frac{V(x_i+a) - 2V(x_i) + V(x_i-a)}{a^2} = 0$$

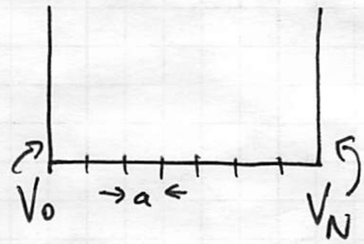
Re writing this equation shows us precisely what we expect ~~for~~ from GP #3:

$$V(x_i) = \frac{1}{2}\left[V(x_i+a) + V(x_i-a)\right]$$

the potential at x_i is equal to the average of the potential at x_i+a and x_i-a. Note a can be reduced for better resolution.

Making this a computational task

Consiter a 1D Laplace Eqn problem,



with set Boundary Conditions,

$V(x_0) = V_0$

$V(x_N) = V_N$

We will cut up the space, called a "mesh", in equal size chunks, a.

Pseudcode:

let $V(x_i)$ = some reasnoable random #'s. (e.g. between V_0 & V_N)

then,

for i in range (1, N-1)

$$V(x_i) = \frac{1}{2}\left[V(x_i - a) + V(x_i + a)\right]$$

this completes one execution,

Continue until error converges to selected error <u>or</u> to max number of iterations.

Convergence, when error is acceptable.

for kth run

$$\text{Error} = V_{k+1} - V_k$$

$$\max(\text{Error}) < \text{acceptance?}$$

Stop

else run again

just one possible way.

Let's go back to the 2D problem,

$$\nabla^2 V(x,y) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

We can develop an approximate form for this PDE, by approximating the derivative,

$$\frac{\partial^2 V}{\partial x^2} = \frac{V(x+a,y) - 2V(x,y) + V(x-a,y)}{a^2}$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{V(x,y+a) - 2V(x,y) + V(x,y-a)}{a^2}$$

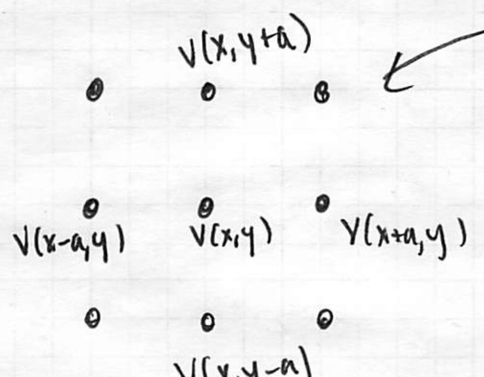
so that,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{V(x+a,y) + V(x,y+a) + V(x-a,y) + V(x,y-a) - 4V(x,y)}{a^2}$$

B/c $\nabla^2 V(x,y) = 0$,

$$V(x,y) = \frac{1}{4}\left[V(x+a,y) + V(x,y+a) + V(x-a,y) + V(x,y-a)\right]$$

V @ x,y is average of surrounding pts!



these pts. we call the mesh.

to find V(x,y), we successively average over the surrounding mesh pts.