Chapter 2 Free Oscillations of Systems with Many Degrees of Freedom

2.1 Introduction

In Chap. 1 we studied oscillations of systems having one or two degrees of freedom. In this chapter we shall study systems having N degrees of freedom, where N can range up to some very large number, which we shall loosely call "infinity."

For a system with N degrees of freedom, there are always exactly N modes (see Prob. 1.17). Each mode has its own frequency ω and its own "shape" given by the amplitude ratios $A:B:C:D:\ldots$ etc., corresponding to the degrees of freedom $a, b, c, d, \ldots$, etc. In each mode, all moving parts go through their equilibrium positions simultaneously; that is, every degree of freedom oscillates in that mode with the same phase constant. Thus there is a single phase constant for the entire mode, which is determined by the initial conditions. Since each degree of freedom oscillates in a given mode with the same frequency ω, each moving part experiences the same return force per unit displacement per unit mass, given by ω^2.

As an example, suppose we have a system with four degrees of freedom a, b, c, d. Then there are four modes. Suppose that in mode 1 the amplitude ratios are

$$A:B:C:D = 1:0:-2:7.$$

Then the motions of a, b, c, and d (if mode 1 is the only excited mode) are given by

$$\psi_a = A_1 \cos(\omega_1 t + \varphi_1), \qquad \psi_b = 0, \qquad \psi_c = -2\psi_a, \qquad \psi_d = 7\psi_a,$$

where A_1 and φ_1 depend on the initial conditions.

If a system contains a very large number of moving parts, and if these parts are distributed within a limited region of space, the average distance between neighboring moving parts becomes very small. As an approximation, one may wish to think of the number of parts as becoming infinite and the distance between neighboring parts as going to zero. One then says that the system behaves as if it were "continuous." Implicit in this point of view is the assumption that the motion of near neighbors is nearly the same. This assumption allows us to describe the vector displacement of all the moving parts in a small neighborhood of a point x, y, z, with a single vector quantity $\boldsymbol{\psi}(x,y,z,t)$. Then the "displacement" $\boldsymbol{\psi}(x,y,z,t)$ is a continuous function of position, x, y, z, and of time t. It replaces the description using the displacements $\psi_a(t)$, $\psi_b(t)$, etc., of the individual parts. We then say we are dealing with *waves.*

Standing waves are normal modes. The modes of a continuous system are called *standing waves,* or *normal modes,* or simply *modes.* According to the discussion above, a truly continuous system has an infinite number of independent moving parts, although they occupy a finite space. There are therefore an infinite number of degrees of freedom, and hence an infinite number of modes. This is not literally true for a real material system. One liter of air does not contain an infinite number of moving parts, but only 2.7×10^{22} molecules, each of which has three degrees of freedom (for motion along x, y, and z directions). Thus a bottle containing 1 liter of air does not have an infinite number of possible vibrational modes of the air, but only 8×10^{22} at most. Anyone who has practiced blowing a bottle or a flute knows that it is not easy to excite more than the first few modes. (We usually distinguish the modes by calling the one with the lowest frequency number 1, the next higher number 2, etc.) In practice we are often concerned only with the first few (or few dozen or few thousand) modes. As we shall see, it turns out that the lowest modes behave as if the system were continuous.

The most general motion of a system can be written as a superposition of all its modes, with the amplitude and phase constant of each mode set by the initial conditions. The appearance of the vibrating system in such a general situation is very complicated, simply because the eye and brain cannot contemplate several things at once without confusion. It is not easy to look at the complete motion and "see" each mode separately when many are present.

Modes of beaded string. We study first the transverse oscillations of beaded strings. By "strings" we shall really mean springs. We will assume that we have linear (i.e., Hooke's law) massless springs connecting point masses M. (In our figures, we will draw the springs as straight lines rather than as helices.)

In Fig. 2.1 we exhibit a sequence of systems of beaded strings. The first system has $N = 1$ (one degree of freedom), the next $N = 2$, etc. In each case, we exhibit without proof the configurations of the normal modes. Later we shall derive the exact configuration and frequency for each mode.

It should already be possible for you to see (assuming the configurations shown are those of the modes) that we have correctly ordered the configurations in order of ascending mode frequency. That is because the strings

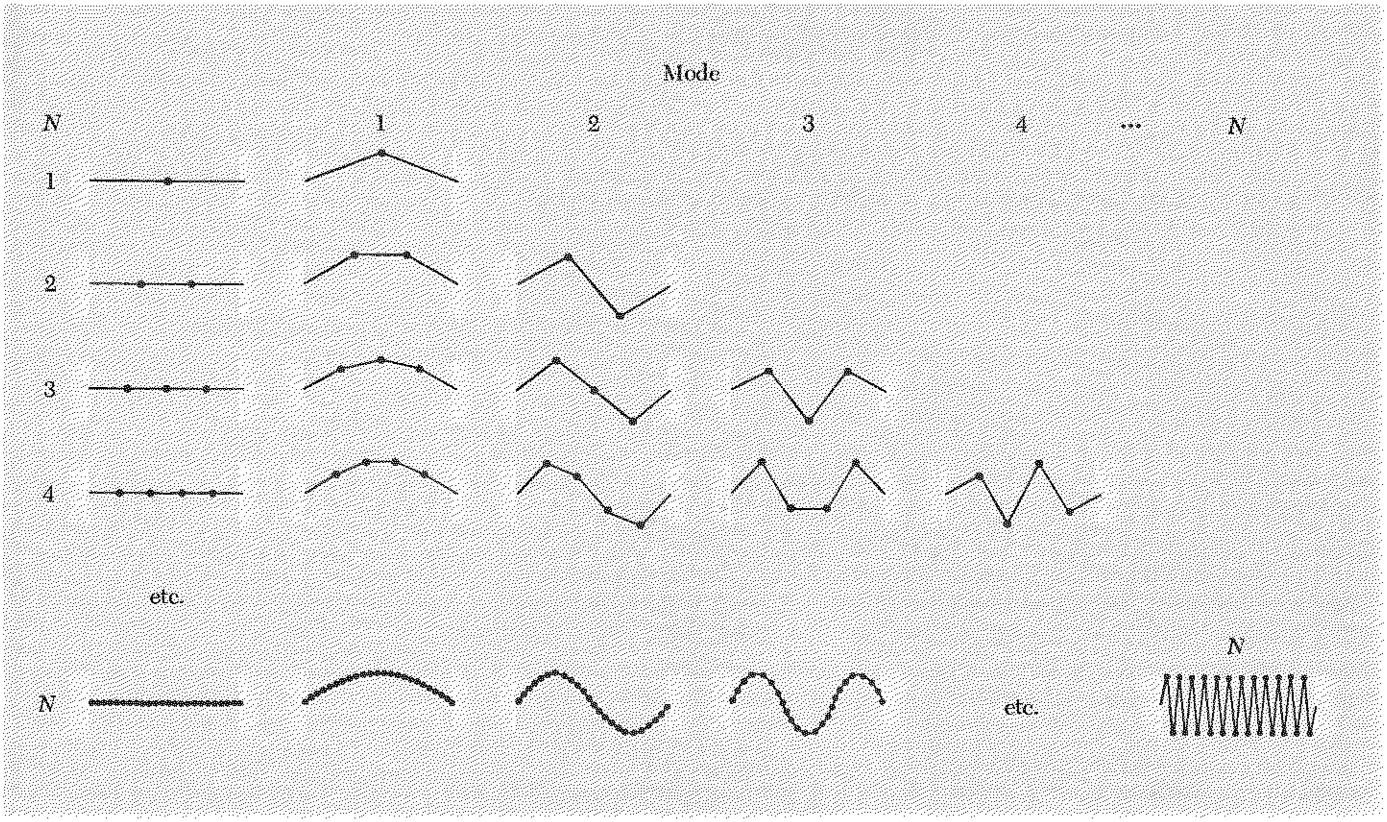

Fig. 2.1 Transverse vibrational modes of a beaded string. A string with N beads has N modes. In mode m the string crosses the equilibrium axis m − 1 times and has m half-wavelengths. The highest frequency mode is the "zig-zag" configuration shown.

make increasingly large angles with the equilibrium axis as we increase the mode number (taking the displacement of a given bead to be the same). Consequently the return force per unit displacement per unit mass for a given bead in a given system increases when we go from one configuration to the next, and therefore so does the mode frequency.

Another thing that is apparent is that our sequence of assumed mode shapes always gives exactly N configurations: the first mode always has zero "nodes" (places where the string crosses the axis, excluding the end points), the second has one node, etc. The highest mode always has the largest possible number of nodes, namely $N - 1$, which is achieved by "zig-zagging" up and down, i.e., crossing the axis once between each two successive masses.

2.2 *Transverse Modes of Continuous String*

We now consider the case where N is huge, say $N = 1{,}000{,}000$ or so. Then for the lowest modes (say the first few thousand), there are a very large number of beads between each node. Thus the displacement varies slowly from one bead to the next. [We shall *not* consider here the highest modes, since they approach the "zigzag limit," where a description using a continuous function $\psi(x,y,z,t)$ is not possible.] Therefore, in accordance with the remarks above, we shall not describe the instantaneous configuration by the list of displacements $\psi_a(t)$, $\psi_b(t)$, $\psi_c(t)$, $\psi_d(t)$, etc., of each bead. Instead we consider all the particles with *equilibrium* positions in the neighborhood of the point x,y,z (a neighborhood being an infinitesimal cube, if you wish, with edges of length Δx, Δy, and Δz) as having the same instantaneous vector displacement $\boldsymbol{\psi}(x,y,z,t)$:

$$\boldsymbol{\psi}(x,y,z,t) = \hat{\mathbf{x}}\psi_x(x,y,z,t) + \hat{\mathbf{y}}\psi_y(x,y,z,t) + \hat{\mathbf{z}}\psi_z(x,y,z,t), \tag{1}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors and ψ_x, ψ_y, and ψ_z are the components of the vector displacement $\boldsymbol{\psi}$. It is important to realize that x,y,z label the *equilibrium* position of the particles in that neighborhood. Thus x,y,z are not functions of time.

Longitudinal and transverse vibration. Equation (1) is of a much more general form than we need in order to study the vibrations of a string. Suppose that at equilibrium the string is stretched along the z axis. Then the coordinate z is sufficient to label the equilibrium position of each bead (to an accuracy Δz) and Eq. (1) can be written in the simpler form

$$\boldsymbol{\psi}(z,t) = \hat{\mathbf{x}}\psi_x(z,t) + \hat{\mathbf{y}}\psi_y(z,t) + \hat{\mathbf{z}}\psi_z(z,t). \tag{2}$$

Vibrations along the z direction are called *longitudinal* vibrations. Vibrations along the x and y directions are called *transverse* vibrations. At present we wish to consider only the transverse vibrations of the string. Therefore we assume ψ_z is zero:

$$\boldsymbol{\psi}(z,t) = \hat{\mathbf{x}}\psi_x(z,t) + \hat{\mathbf{y}}\psi_y(z,t). \tag{3}$$

Linear polarization. As a further simplification, we assume that the vibrations are entirely along $\hat{\mathbf{x}}$ (i.e., $\psi_y \equiv 0$). The vibrations are then said to be *linearly polarized* along $\hat{\mathbf{x}}$. (In Chap. 8 we shall study general states of polarization.) Now we can drop the unit vector $\hat{\mathbf{x}}$ and the subscript on ψ_x from the notation:

$$\psi(z,t) = \text{instantaneous transverse displacement of particles having equilibrium position } z. \tag{4}$$

Now consider a very small segment of the continuous string. At equilibrium, the segment occupies a small interval of length Δz centered at z. The mass ΔM of the segment divided by the length Δz is defined as the *mass density* ρ_0, measured in units of mass per unit length:

$$\Delta M = \rho_0\,\Delta z. \tag{5}$$

The mass density is assumed to be uniform along the string. The string tension at equilibrium, denoted by T_0, is also assumed to be uniform.

For a general (nonequilibrium) situation, the segment has a transverse displacement $\psi(z,t)$, averaged over the segment. (See Fig. 2.2.) The segment is no longer exactly straight; it has (generally) a slight curvature. This is indicated in Fig. 2.2 by the fact that θ_1 and θ_2 are not equal. The tension in the segment is no longer T_0, since the segment is longer than its equilibrium length Δz. Let us find the net force F_x on the segment at the instant shown. At its left end, the segment is pulled downward with a force $T_1 \sin \theta_1$. At its right end, it is pulled upward with a force $T_2 \sin \theta_2$. Thus the net force upward is

$$F_x(t) = T_2 \sin \theta_2 - T_1 \sin \theta_1. \tag{6}$$

We want to express $F_x(t)$ in terms of $\psi(z,t)$ and its space derivative

$$\frac{\partial \psi(z,t)}{\partial z} = \text{slope of string at position } z \text{ at time } t. \tag{7}$$

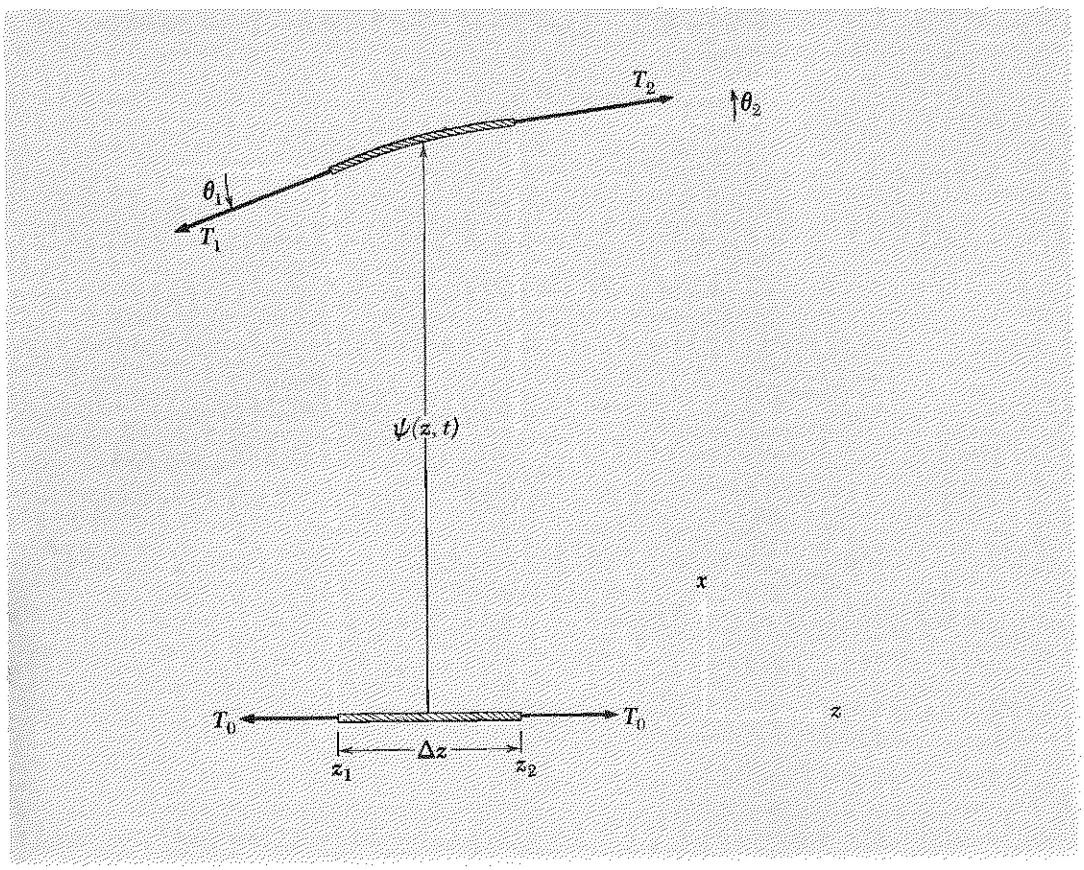

Fig. 2.2 Transverse oscillations of a continuous string. At bottom is the equilibrium position of an infinitesimal segment along the z axis. Above is a general position and configuration of the same segment.

According to Fig. 2.2, the string slope at z_1 is $\tan\theta_1$, and the slope at z_2 is $\tan\theta_2$. Also, $T_1\cos\theta_1$ is the horizontal component of the string tension at z_1, and $T_2\cos\theta_2$ is the horizontal component at z_2. Now, we want eventually to obtain a *linear* differential equation of motion. To this end, we shall assume that we can use either the slinky approximation or the small-oscillations approximation. In the slinky approximation, T is larger than T_0 by a factor $1/\cos\theta$, because the segment is longer than Δz by a factor $1/\cos\theta$. Therefore $T\cos\theta = T_0$. In the small-oscillations approximation, we neglect the increase in length of the segment, and we also approximate $\cos\theta$ by 1. Thus we have $T\cos\theta = T_0$ in that case also. Then Eq. (6) gives

$$\begin{aligned} F_z(t) &= T_2\sin\theta_2 - T_1\sin\theta_1 \\ &= T_2\cos\theta_2\tan\theta_2 - T_1\cos\theta_1\tan\theta_1 \\ &= T_0\tan\theta_2 - T_0\tan\theta_1 \\ &= T_0\left(\frac{\partial\psi}{\partial z}\right)_2 - T_0\left(\frac{\partial\psi}{\partial z}\right)_1. \end{aligned} \tag{8}$$

Now consider the function $f(z)$ defined by

$$f(z) = \frac{\partial\psi(z,t)}{\partial z}, \tag{9}$$

where we have suppressed the variable t in writing $f(z)$ because we intend to hold t constant. We expand $f(z)$ in a Taylor's series around z_1 and then set $z = z_2$. [See Appendix Eq. (3)]:

$$f(z_2) = f(z_1) + (z_2 - z_1)\left(\frac{df}{dz}\right)_1 + \frac{1}{2}(z_2 - z_1)^2\left(\frac{d^2f}{dz^2}\right)_1 + \cdots, \tag{10}$$

where $z_2 - z_1 = \Delta z$, according to Fig. 2.2. We now go to the limit in which Δz is small enough so that we can neglect quadratic and higher terms in Eq. (10). Then we write

$$\begin{aligned} f(z_2) - f(z_1) = \Delta z\left(\frac{df}{dz}\right)_1 &= \Delta z\frac{d}{dz}\left(\frac{\partial\psi(z,t)}{\partial z}\right) \\ &= \Delta z\frac{\partial}{\partial z}\left(\frac{\partial\psi(z,t)}{\partial z}\right) \\ &= \Delta z\frac{\partial^2\psi(z,t)}{\partial z^2}. \end{aligned} \tag{11}$$

Notice that in arriving at Eq. (11) we dropped the subscript 1. That is because it does not matter where in the interval Δz we evaluate the z derivative, since we are neglecting higher derivatives in the Taylor's series, Eq. (10). Notice also that we must write the space derivative as a partial derivative once we use the notation $\psi(z,t)$.

We can now use Eqs. (9) and (11) in Eq. (8) to obtain for the net force on the segment the result

$$F_x(t) = T_0\,\Delta_z \frac{\partial^2\psi(z,t)}{\partial z^2}. \tag{12}$$

We now use Newton's second law. The force F_x, as given by Eq. (12), equals the mass ΔM of the segment times the acceleration of the segment. The velocity and acceleration of the segment with equilibrium position z are expressed in terms of $\psi(z,t)$ and its derivatives as follows:

$$\begin{aligned} \psi(z,t) &= \text{displacement} \\ \frac{\partial\psi(z,t)}{\partial t} &= \text{velocity} \\ \frac{\partial^2\psi(z,t)}{\partial t^2} &= \text{acceleration.} \end{aligned} \tag{13}$$

Thus Newton's law [with $\Delta M = \rho_0\,\Delta z$] gives

$$\rho_0\,\Delta z\frac{\partial^2\psi}{\partial t^2} = F_x = T_0\,\Delta z\frac{\partial^2\psi}{\partial z^2},$$

i.e.,

$$\boxed{\frac{\partial^2\psi(z,t)}{\partial t^2} = \frac{T_0}{\rho_0}\frac{\partial^2\psi(z,t)}{\partial z^2}} \tag{14}$$

Classical wave equation. Equation (14) is a very famous second-order linear partial differential equation. It is called the *classical wave equation.* We will encounter it often and will eventually know many of the properties of its solutions and the physical situations where it occurs. (Of course the positive constant T_0/ρ_0 is special to the string. In other physical applications, some other positive constant appears in its place in the wave equation.)

Standing waves. We are trying to find the normal modes—the standing waves—of a continuous string. Therefore we *assume* that we have a mode. We assume that all parts of the string oscillate in harmonic motion at the same angular frequency ω and with the same phase constant φ. Thus $\psi(z,t)$, which is the displacement of string particles with equilibrium position z, should have the same time dependence, $\cos(\omega t + \varphi)$ for all particles, i.e., for all z. As usual, the phase constant φ corresponds to the "turn-on time" of the mode. The "shape" of a mode made up of discrete degrees of freedom labeled a, b, c, etc., is given by the relative vibration amplitudes, A, B, C, etc. In the present case of a continuous string, where the (infinitely many) degrees of freedom are labeled by the parameter z, the amplitude of vibration of the degrees of freedom at z (i.e., in a small neighborhood of z) can be written as a continuous function of z denoted by $A(z)$. The shape of $A(z)$ as a function of z depends on the mode; that is, each mode has a different $A(z)$. Thus we can write down the *general expression for a standing wave:*

$$\psi(z,t) = A(z)\cos(\omega t + \varphi). \tag{15}$$

The acceleration corresponding to Eq. (15) is

$$\frac{\partial^2\psi}{\partial t^2} = -\omega^2\psi = -\omega^2 A(z)\cos(\omega t + \varphi) \tag{16}$$

The second partial derivative of Eq. (15) with respect to z is

$$\begin{aligned}\frac{\partial^2\psi}{\partial z^2} &= \frac{\partial^2[A(z)\cos(\omega t + \varphi)]}{\partial z^2} \\ &= \cos(\omega t + \varphi)\frac{d^2A(z)}{dz^2},\end{aligned} \tag{17}$$

where we have an ordinary derivative with respect to z rather than a partial derivative because $A(z)$ has no time dependence. Inserting Eqs. (16) and (17) into Eq. (14) and canceling the common factor $\cos(\omega t + \varphi)$, we obtain

$$\frac{d^2A(z)}{dz^2} = -\omega^2\frac{\rho_0}{T_0}A(z). \tag{18}$$

Equation (18) governs the shape of the mode. Since each mode has a different angular frequency ω, and since ω^2 appears in Eq. (18), we see that different modes have different shapes, as expected.

Equation (18) is of the form of the differential equation for harmonic oscillation, but for oscillation in space rather than in time. The general form of a harmonic oscillation in space can be written

$$A(z) = A\sin\left(2\pi\frac{z}{\lambda}\right) + B\cos\left(2\pi\frac{z}{\lambda}\right), \tag{19}$$

where the constant λ represents the distance over which one complete oscillation occurs. Thus it is called the *wavelength*. It is the parameter for oscillations in space analogous to the period T for oscillations in time. The wavelength λ is measured in units of centimeters per cycle (i.e., per cycle of spatial oscillation along z), or simply in centimeters.

To see how to adapt this solution to Eq. (18), differentiate Eq. (19) twice:

$$\frac{d^2A(z)}{dz^2} = -\left(\frac{2\pi}{\lambda}\right)^2 A(z). \tag{20}$$

Then comparing Eqs. (18) and (20), we see that we need to have

$$\left(\frac{2\pi}{\lambda}\right)^2 = \omega^2\left(\frac{\rho_0}{T_0}\right) = (2\pi\nu)^2\frac{\rho_0}{T_0}, \tag{21}$$

i.e.,

$$\boxed{\lambda\nu = \sqrt{\frac{T_0}{\rho_0}} \equiv v_0 = \text{constant.}} \tag{22}$$

Wave velocity. Equation (22) gives the relation between wavelength and frequency for transverse standing waves on a continuous homogeneous string. The constant $(T_0/\rho_0)^{1/2}$ has the dimensions of velocity, since $\lambda\nu$ has dimensions length/time. The velocity $v_0 \equiv (T_0/\rho_0)^{1/2}$ is called the "phase velocity for traveling waves," for this system. (We will study traveling waves in Chap. 4.) In our present study of standing waves, the concept of phase velocity is not needed, because standing waves do not "go anywhere." They "stand and wave" like a big "distributed" harmonic oscillator. Hereafter in this chapter we shall avoid calling $(T_0/\rho_0)^{1/2}$ a velocity, because we want your mental picture to be that of standing waves.

The general solution for the displacement $\psi(z,t)$ of the string in a single mode (standing wave) is obtained by combining Eqs. (15) and (19):

$$\psi(z,t) = \cos(\omega t + \varphi)[A\sin(2\pi z/\lambda) + B\cos(2\pi z/\lambda)]. \tag{23}$$

Boundary conditions. Equation (23) is slightly *too* general. It does not manifest the important boundary conditions. Our vibrating string is *fixed at both ends,* but we have not yet incorporated that bit of information into the solution. We do so as follows. Suppose the string has total length L. Let us choose the origin of coordinates so that the left-hand end of the string is at $z = 0$. The right-hand end is then at $z = L$. Consider $z = 0$. The string is fixed there, so $\psi(0,t)$ must be zero for all t. This condition requires that $B = 0$, since, for all times t,

$$\psi(0,t) = \cos(\omega t + \varphi)[0 + B] = 0. \tag{24}$$

Thus we have

$$\psi(z,t) = A\cos(\omega t + \varphi)\sin\frac{2\pi z}{\lambda}. \tag{25}$$

The other boundary condition is that the string be fixed at $z = L$, so $\psi(L,t)$ must be zero for all t. We certainly do not want to choose $A = 0$ in Eq. (25), since that corresponds to the uninteresting situation of a string permanently at rest. The only way we can satisfy the boundary condition at L is to have

$$\sin \frac{2\pi L}{\lambda} = 0. \tag{26}$$

The only wavelengths λ that can satisfy this boundary condition are those for which the number of half-wavelengths, L, is an integer. Thus the acceptable wavelengths must satisfy one of the following possibilities:

$$\frac{2\pi L}{\lambda} = \pi, 2\pi, 3\pi, 4\pi, 5\pi, \ldots . \tag{27}$$

(Why did we exclude the case $2\pi L/\lambda = 0$?) This sequence of possible ways to satisfy the boundary conditions corresponds to all the possible modes of the string. We number the modes according to the sequence, beginning with the first term in the sequence as number 1. Then according to Eq. (27), we have the wavelengths of the modes given by

$$\lambda_1 = 2L, \qquad \lambda_2 = \tfrac{1}{2}\lambda_1, \qquad \lambda_3 = \tfrac{1}{3}\lambda_1, \qquad \lambda_4 = \tfrac{1}{4}\lambda_1, \qquad \ldots . \tag{28}$$

Harmonic frequency ratios. The corresponding frequencies of the modes are found by using Eq. (22):

$$\nu_1 = \frac{v_0}{\lambda_1}, \qquad \nu_2 = 2\nu_1, \qquad \nu_3 = 3\nu_1, \qquad \nu_4 = 4\nu_1, \qquad \ldots . \tag{29}$$

The frequencies $2\nu_1$, $3\nu_1$, etc., are called the second, third, etc., *harmonics* of the *fundamental* frequency ν_1. The fact that the mode frequencies ν_2, ν_3, etc., consist of a sequence of *harmonics* of the lowest mode frequency ν_1 is a result of our assumption that the string is perfectly uniform and flexible. Most real physical systems have mode frequencies that do not follow this harmonic sequence of frequency ratios. For example, the mode frequencies for a string of nonuniform mass density do not form a sequence of harmonics of the fundamental. Instead one might have, for example, $\nu_2 = 2.78\nu_1$, $\nu_3 = 4.62\nu_1$, etc. For a real piano or violin string, the mode requencies follow approximately, but not exactly, the harmonic sequence. That is because they are not perfectly flexible. (For a qualitative argument that shows how these "harmonic" frequency ratios are due to the uniformity of the string, see Prob. 2.7.)

The modes of the string are shown in Fig. 2.3. The equilibrium configuration would correspond to the missing first term, $2\pi L/\lambda = 0$, in the sequence given by Eq. (27). The corresponding frequency is zero. There is no motion, and the equilibrium state is not called a mode.

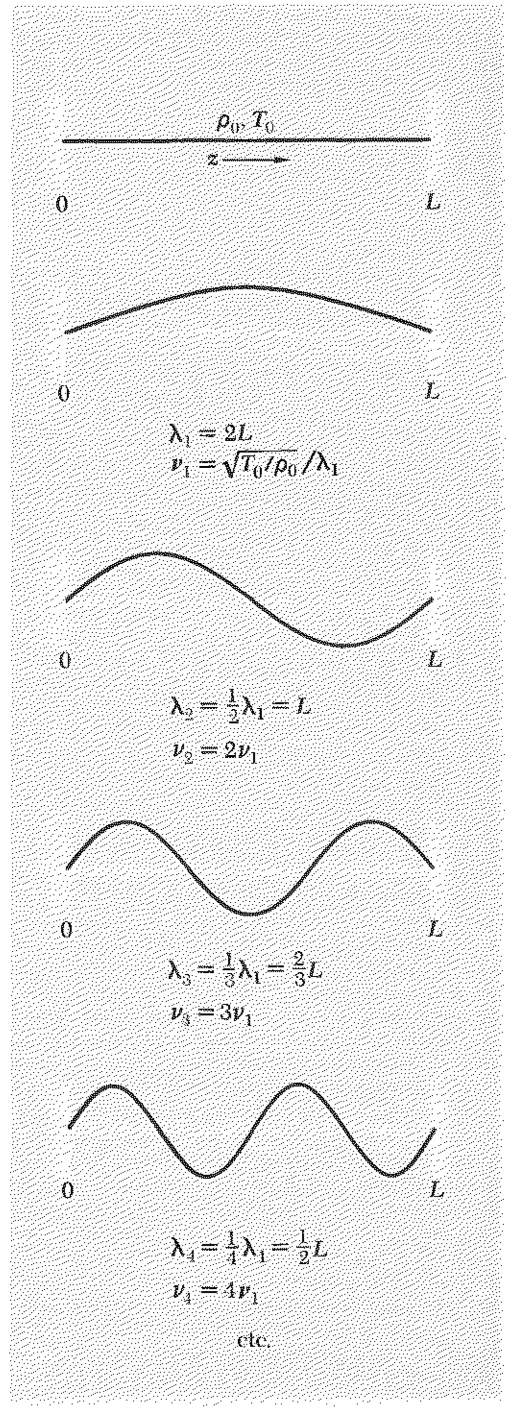

Fig. 2.3 Modes of continuous homogeneous string with fixed ends.

Wavenumber. The inverse of the wavelength λ is called the *wavenumber* σ. Its units are cycles per centimeter or, more often, "inverse centimeters." It is the parameter for oscillations in space analogous to the frequency ν for oscillations in time.

$$\sigma = \frac{1}{\lambda} = \text{wavenumber (cycles per cm).} \tag{30}$$

The wavenumber times 2π is called the *angular wavenumber* k. Its units are radians of phase per centimeter. It is the quantity for oscillations in space analogous to the angular frequency ω for oscillations in time.

$$k = \frac{2\pi}{\lambda} = \text{angular wavenumber (radians per cm).} \tag{31}$$

We can illustrate the use of these quantities by writing the same standing wave in several equivalent forms:

$$\psi(z,t) = A \sin 2\pi \frac{t}{T} \sin 2\pi \frac{z}{\lambda} = A \sin 2\pi\nu t \sin 2\pi\sigma z = A \sin \omega t \sin kz. \tag{32}$$

As another illustration, we can describe the sequence of normal modes given by Eqs. (27), (28), and (29) as follows:

$$k_1 L = \pi \text{ rad,} \qquad k_2 L = 2\pi \text{ rad,} \qquad k_3 L = 3\pi \text{ rad,} \qquad \text{etc.} \tag{33}$$

$$\sigma_1 L = \tfrac{1}{2} \text{ cycle,} \qquad \sigma_2 L = 1 \text{ cycle,} \qquad \sigma_3 L = \tfrac{3}{2} \text{ cycle,} \qquad \text{etc.} \tag{34}$$

Dispersion relation. Equation (22) gives the relation between frequency and wavelength for the normal modes of the uniform flexible string:

$$\nu = \sqrt{\frac{T_0}{\rho_0}} \cdot \frac{1}{\lambda} = \sqrt{\frac{T_0}{\rho_0}} \cdot \sigma,$$

or (multiplying by 2π)

$$\omega = \sqrt{\frac{T_0}{\rho_0}}\, k. \tag{35}$$

Equation (35) gives the relation between frequency and wavenumber for the normal modes of the string. (Note that we dropped the adjective "angular" from the designations "angular frequency" and "angular wavenumber." This is common practice, but the symbols and the units always remove any ambiguity.) Such a relation, giving ω as a function of k, is called a *dispersion relation*. It is a convenient way of characterizing the wave behavior of a system.

Dispersion law for real piano string. The dispersion relation given by Eq. (35) is extremely simple, but we shall find more complicated ones later. For a more complicated dispersion relation, the quantity $\lambda\nu = \omega/k$ is *not* constant, i.e., it is not independent of wavelength. For example, it turns out that the dispersion law for a real piano string is given approximately by

$$\frac{\omega^2}{k^2} = \frac{T_0}{\rho_0} + \alpha k^2 \tag{36}$$

where α is a small positive constant that would be zero if the string were perfectly flexible. [In that case Eq. (36) reduces to Eq. (35).] The modes of a real piano string have the same spatial dependence as those of a perfectly flexible string, i.e., $\lambda_1 = 2L$, $\lambda_2 = \frac{1}{2}\lambda_1$, $\lambda_3 = \frac{1}{3}\lambda_1$, etc., because the boundary conditions are the same. But the mode frequencies do *not* satisfy the "harmonic" sequence $\nu_2 = 2\nu_1$, $\nu_3 = 3\nu_1$, etc., because the dispersion relation Eq. (36) does not give that sequence. The harmonic sequence is obtained only in the idealized limit where α is zero, i.e., where we have $\lambda\nu$ = constant. For a real piano string the frequencies of the higher modes are slightly "sharper" (i.e., have slightly higher frequencies) than the frequencies given by the harmonic sequence.

Nondispersive and dispersive waves. Waves satisfying the simple dispersion relation ω/k = constant are called "nondispersive waves." When ω/k depends on the wavelength (and hence on the frequency), the waves are called "dispersive." For dispersive waves, it is customary to make a plot of ω versus k. In the present example of the flexible string this plot is just a straight line passing through the point $\omega = k = 0$ and having slope $(T_0/\rho_0)^{1/2}$, as shown in Fig. 2.4.

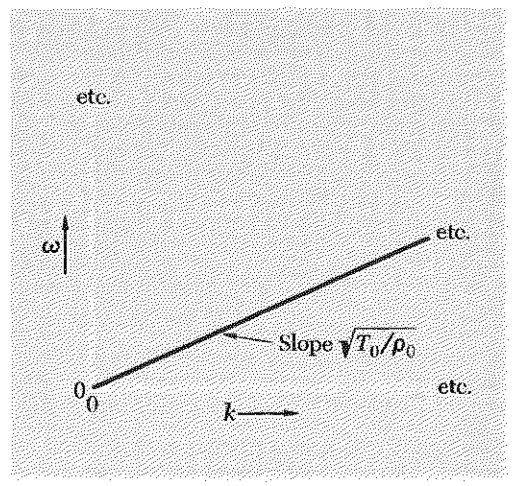

Fig. 2.4 ***Dispersion relation for continuous, homogeneous, flexible string.***

2.3 *General Motion of Continuous String and Fourier Analysis*

The most general state of motion of the continuous string (with both ends fixed and for transverse vibrations along x) is given by a superposition of all the modes, numbered 1, 2, 3, . . . , with amplitudes $A_1, A_2, A_3, \ldots$, and phase constants $\varphi_1, \varphi_2, \varphi_3, \ldots$:

$$\psi(z,t) = A_1 \sin k_1 z \cos(\omega_1 t + \varphi_1) + A_2 \sin k_2 z \cos(\omega_2 t + \varphi_2) + \cdots, \tag{37}$$

where k_n are chosen as described in the preceding section to satisfy the boundary conditions at $z = 0$ and $z = L$, and where ω_n are related to k_n by the dispersion relation $\omega(k)$. The amplitudes A_n and phase constants φ_n, which complete the description of the motion for all positions z and times t, are determined by specifying the *initial conditions,* namely, the instantaneous displacement $\psi(z,t)$ and the corresponding instantaneous velocity $v(z,t) = \partial\psi(z,t)/\partial t$ for each point z at $t = 0$.

Motion of string fixed at both ends. Suppose that for $t < 0$ we constrain the string to follow a prescribed shape $f(z)$ by means of some sort of template. Then, at $t = 0$, we let the string go by suddenly removing the template. Thus at $t = 0$ each part of the string has its displacement $\psi(z,0)$ equal to $f(z)$ and has velocity $v(z,0)$ equal to zero. Now, the nth term in the velocity [which is the time derivative of Eq. (37)] is proportional to $\sin(\omega_n t + \varphi_n)$, which reduces to $\sin \varphi_n$ at $t = 0$. Thus we can make $v(z,0) = 0$ for all z simply by setting each phase constant φ_n equal either to zero or to π. However, the phase constant $\varphi_1 = \pi$ (for example) is just equivalent to a minus sign affixed to A_1. Therefore we can satisfy these initial conditions if we set all the phase constants to zero but allow the amplitudes A_1, A_2, etc., to be either positive or negative. Thus we have, for $v(z,0) = 0$,

$$\psi(z,t) = A_1 \sin k_1 z \cos \omega_1 t + A_2 \sin k_2 z \cos \omega_2 t + \cdots, \tag{38}$$

and, at $t = 0$,

$$\psi(z,0) = f(z) = A_1 \sin k_1 z + A_2 \sin k_2 z + \cdots. \tag{39}$$

As we shall see below, Eq. (39) determines the amplitudes $A_1, A_2, \ldots$.

Fourier series for function with zeros at both ends. Now, the function $f(z)$ can be a very general function of z. The only condition we specified was that it was to constrain the string. Therefore, virtually all we require of $f(z)$ is that we have $f(z) = 0$ at $z = 0$ and $z = L$. We also require that $f(z)$ not be "jagged" on a "small" scale, since our wave function $\psi(z,t)$ is supposed to be a slowly varying function of z. Therefore, $f(z)$ must be reasonably smooth in order for us to be able to use it to constrain the string and still have the string obey the differential equation that we obtained in the "continuous" approximation. Thus we have found that *any* reasonable function $f(z)$ that vanishes at $z = 0$ and L can be expanded in a series of the form of Eq. (39), i.e., as a sum of sinusoidal oscillations. Equation (39) is called a *Fourier series* or *Fourier expansion*. It is a special example of a Fourier series in that it applies only to functions $f(z)$ that vanish at $z = 0$ and L. However, a much broader class of functions can be expressed in appropriate Fourier expansions. We shall now find this broader class of functions.

Our function $f(z)$ was used to constrain the string, and therefore it was defined only between $z = 0$ and L. However, the functions $\sin k_1 z$, $\sin 2k_1 z$, $\sin 3k_1 z$, etc., that make up the infinite series of Eq. (39) are defined for all z from $-\infty$ to $+\infty$. Also, we notice that $\sin k_1 z$ is *periodic* in z with period λ_1. This means it satisfies the *periodicity condition*, namely, that for any given z, it must have the same value at $z + \lambda_1$ as it does at z. (The period λ_1 is $2L$ in our example.) We notice that the function $\sin 2k_1 z$ is also periodic in z with period λ_1. (Of course it goes through two cycles in distance λ_1; it is thus periodic with period $\frac{1}{2}\lambda_1$, as well as periodic with

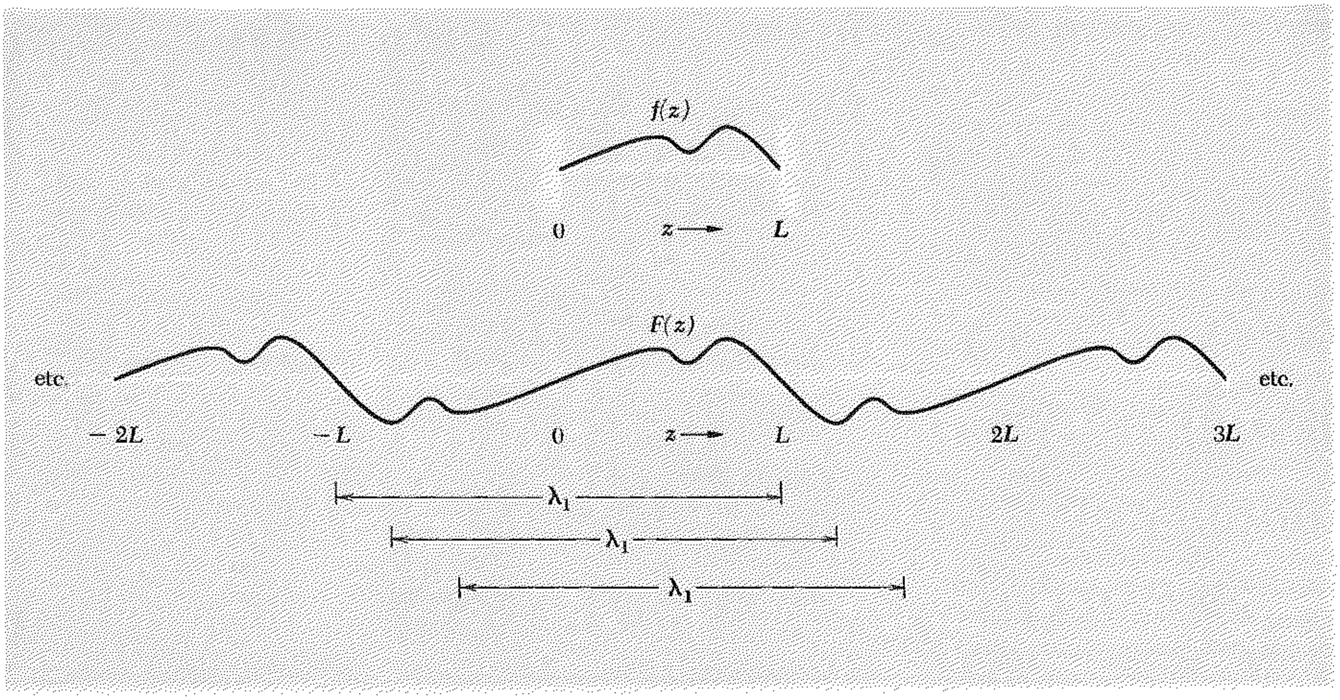

Fig. 2.5 Construction of a periodic function $F(z)$ with period $\lambda_1 = 2L$ from a function $f(z)$ that vanishes at $z = 0$ and L. Note that $F(z)$ satisfies the periodicity condition.

period λ_1.) In fact, all the sinusoidal functions in the expansion, Eq. (39), are periodic in z with period λ_1. Therefore, the expansion itself is periodic with period λ_1. Thus we can broaden the class of functions which have a Fourier expansion of the form of Eq. (39): all periodic functions $F(z)$ with period λ_1 that vanish at $z = 0$ and at $z = \frac{1}{2}\lambda_1$ can be expanded in a Fourier series of the form of Eq. (39). Given a function $f(z)$ defined only between $z = 0$ and L and vanishing at those points, we can construct a periodic function $F(z)$ which will have the same Fourier expansion as $f(z)$ by the following procedure: Between $z = 0$ and L, we let $F(z)$ coincide with $f(z)$. Between L and $2L$, we construct $F(z)$ by making an "inverted mirror image" of $f(z)$ in a "mirror" located at $z = L$. Now that we have defined $F(z)$ between $z = 0$ and $2L$, we simply repeat it in successive intervals of length $2L$ to define $F(z)$ for all z. The construction is shown in Fig. 2.5.

Fourier analysis of a periodic function of z. We now broaden the class of functions for which we can write Fourier expansions once more, as follows: Equation (39) corresponds only to functions that are periodic with period λ_1 and that vanish at $z = 0$ and $\frac{1}{2}\lambda_1$. However, the condition that the function vanish at $z = 0$ and $\frac{1}{2}\lambda_1$ was the result of our particular choice of boundary conditions, namely that the string have both ends fixed. Without those particular boundary conditions, we would have obtained solutions for the string vibrations which included not only the terms in $\sin mk_1z$ but also terms in $\cos mk_1z$. These functions are also periodic in z with period λ_1, but they do not vanish at $z = 0$ and $\frac{1}{2}\lambda_1$. (They correspond to string vibrations with a free end or ends.) By including them in the series, we finally arrive at a very general class of functions for which we can write Fourier series: all (reasonable) periodic functions $F(z)$ with period λ_1, i.e., functions such that $F(z + \lambda_1) = F(z)$ for all z, can be expanded in a Fourier series of the form

$$\begin{aligned} F(z) &= \sum_{n=0}^{\infty} \left[A_n \sin n \frac{2\pi}{\lambda_1} z + B_n \cos n \frac{2\pi}{\lambda_1} z \right] \\ &= B_0 + \sum_{n=1}^{\infty} A_n \sin n \frac{2\pi}{\lambda_1} z + \sum_{n=1}^{\infty} B_n \cos n \frac{2\pi}{\lambda_1} z \\ &= B_0 + \sum_{n=1}^{\infty} A_n \sin nk_1 z + \sum_{n=1}^{\infty} B_n \cos nk_1 z. \end{aligned} \tag{40}$$

Finding Fourier coefficients. The process of finding the amplitudes or *Fourier coefficients* B_0, A_n, and B_n (for all n) for a given periodic function $F(z)$ is called *Fourier analysis*. We shall now show you how to find these coefficients.

First we find B_0 as follows: We integrate both sides of Eq. (40) over any complete period of $F(z)$; i.e., we integrate from $z = z_1$ to $z = z_2$, where z_1 is any value of z and where $z_2 = z_1 + \lambda_1$. The function $F(z)$ is assumed to be known; therefore its integral from z_1 to z_2, which is the integral of the left side of Eq. (40), can be found. Now consider the integral of the right side of Eq. (40). There are an infinite number of terms and therefore an infinite number of integrals to consider. The first term is B_0; it produces the integral

$$\int_{z_1}^{z_2} B_0 \, dz = B_0(z_2 - z_1) = B_0\lambda_1. \tag{41}$$

All the other terms give zero when integrated over one period. That is because $\sin nk_1 z$ and $\cos nk_1 z$ are as often negative as positive in any complete period, and therefore they integrate to zero:

$$\int_{z_1}^{z_2} \sin nk_1 z \, dz = 0; \qquad \int_{z_1}^{z_2} \cos nk_1 z \, dz = 0.$$

Thus we have found B_0. It is given by

$$B_0\lambda_1 = \int_{z_1}^{z_2} F(z) \, dz. \tag{42}$$

Next we show you how to find A_m, where m is any particular value of n in Eq. (40) from 1 to infinity. The trick is to multiply both sides of Eq. (40) by $\sin mk_1 z$ and integrate both sides over one complete period of $F(z)$. The integral of the left-hand side can be evaluated since $F(z)$ is known. Now consider the integral of the right-hand side. The first term is the integral of B_0 times $\sin mk_1 z$; that integrates to zero because it includes m complete periods of $\sin mk_1z$. That leaves us with the integrals of $\sin nk_1z \sin mk_1z$ and of $\cos nk_1z \sin mk_1z$ for $n = 1, 2, \ldots$. Consider the particular term that has $n = m$. The square of $\sin mk_1z$ averages to $\frac{1}{2}$ over one period of $F(z)$ of length λ_1 (which is m complete periods of the function $\sin mk_1z$). This gives a contribution $\frac{1}{2}A_m\lambda_1$ to the integral of the right side of Eq. (40). All other terms contribute zero. We see that as follows: Consider for example the integrand $\sin nk_1z \sin mk_1z$, for m not equal to n. This can be written in the form

$$\sin nk_1z \sin mk_1z = \tfrac{1}{2}\cos(n-m)k_1z - \tfrac{1}{2}\cos(n+m)k_1z. \tag{43}$$

Since $n - m$ and $n + m$ are integers, each of the two terms on the right side of Eq. (43) is as often positive as negative in any complete period of $F(z)$ of length λ_1. Therefore both terms integrate to zero (except for the case $n = m$ which we have already considered). Similarly, the terms of the form $\cos nk_1z \sin mk_1z$ integrate to zero because of the identity

$$\cos nk_1z \sin mk_1z = \tfrac{1}{2}\sin(m+n)k_1z + \tfrac{1}{2}\sin(m-n)k_1z.$$

Thus we find that

$$\frac{1}{2}A_m\lambda_1 = \int_{z_1}^{z_2} \sin mk_1z\, F(z)\, dz. \tag{44}$$

Similarly, we can find the coefficients B_m by multiplying both sides of Eq. (40) by $\cos mk_1z$ and integrating over one period of length λ_1. The only nonzero contribution to the integral of the right side comes from the term with coefficient B_m. Thus we find that

$$\frac{1}{2}B_m\lambda_1 = \int_{z_1}^{z_2} \cos mk_1z\, F(z)\, dz. \tag{45}$$

Fourier coefficients. Our results are given by Eqs. (40), (42), (44), and (45), which we collect in one place for convenience of future reference:

$$\boxed{\begin{aligned} F(z) &= B_0 + \sum_{m=1}^{\infty} A_m \sin mk_1z + \sum_{m=1}^{\infty} B_m \cos mk_1z, \\ B_0 &= \frac{1}{\lambda_1}\int_{z_1}^{z_1+\lambda_1} F(z)\, dz, \\ A_m &= \frac{2}{\lambda_1}\int_{z_1}^{z_1+\lambda_1} F(z) \sin mk_1z\, dz, \\ B_m &= \frac{2}{\lambda_1}\int_{z_1}^{z_1+\lambda_1} F(z) \cos mk_1z\, dz, \end{aligned}} \tag{46}$$

where z_1 is any value of z. Equations (46) tell us how to Fourier-analyze $F(z)$, any periodic function of z having period λ_1.

Square wave. Here is an illustrative example, the Fourier analysis of a "square wave." Let $f(z)$ be zero at the points $z = 0$ and $z = L$, but let it equal $+1$ for $0 < z < L$. (This function has a discontinuity at $z = 0$ and another at $z = L$, so that it does not satisfy the assumption in our discussion above that it be "smooth" everywhere. Therefore we cannot reasonably expect the Fourier series to give a perfect representation of a square wave. It turns out that there is a sharp "overshoot spike" at $z = 0$ and at $z = L$ for every partial sum of the series. As more and more terms are added, the spike gets sharper, but its height does not go to zero.)

The periodic function $F(z)$ that we construct according to the prescription of Fig. 2.5 is given as follows: $F(z) = 0$ for $z = 0$; $+1$ for $0 < z < L$; 0 for $z = L$; -1 for $L < z < 2L$; etc.; as shown in Fig. 2.6.

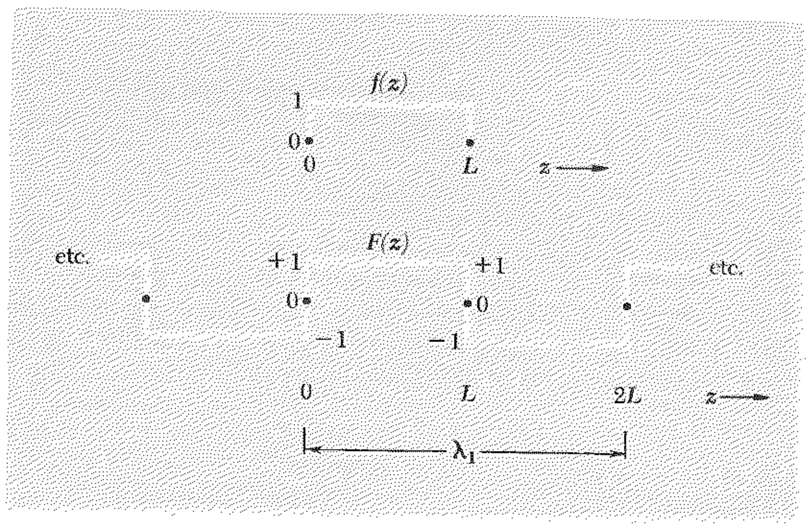

***Fig. 2.6** Square wave f(z). Periodic square wave F(z).*

Using Eqs. (46), one can easily obtain the results (Prob. 2.11) $B_0 = 0$; $B_m = 0$ for all m; $A_m = 0$ for $m = 2, 4, 6, 8 \ldots$ (even integers); $A_m = 4/m\pi$ for $m = 1, 3, 5, 7, \ldots$ (odd integers). Thus $F(z)$ is given by

$$\begin{aligned} F(z) &= B_0 + \sum_{m=1}^{\infty} B_m \cos mk_1z + \sum_{m=1}^{\infty} A_m \sin mk_1z \\ &= \frac{4}{\pi}\left\{\sin k_1z + \frac{1}{3}\sin 3k_1z + \frac{1}{5}\sin 5k_1z + \cdots\right\} \\ &= 1.273 \sin\frac{\pi z}{L} + 0.424 \sin\frac{3\pi z}{L} + 0.255 \sin\frac{5\pi z}{L} + \cdots. \end{aligned} \tag{47}$$

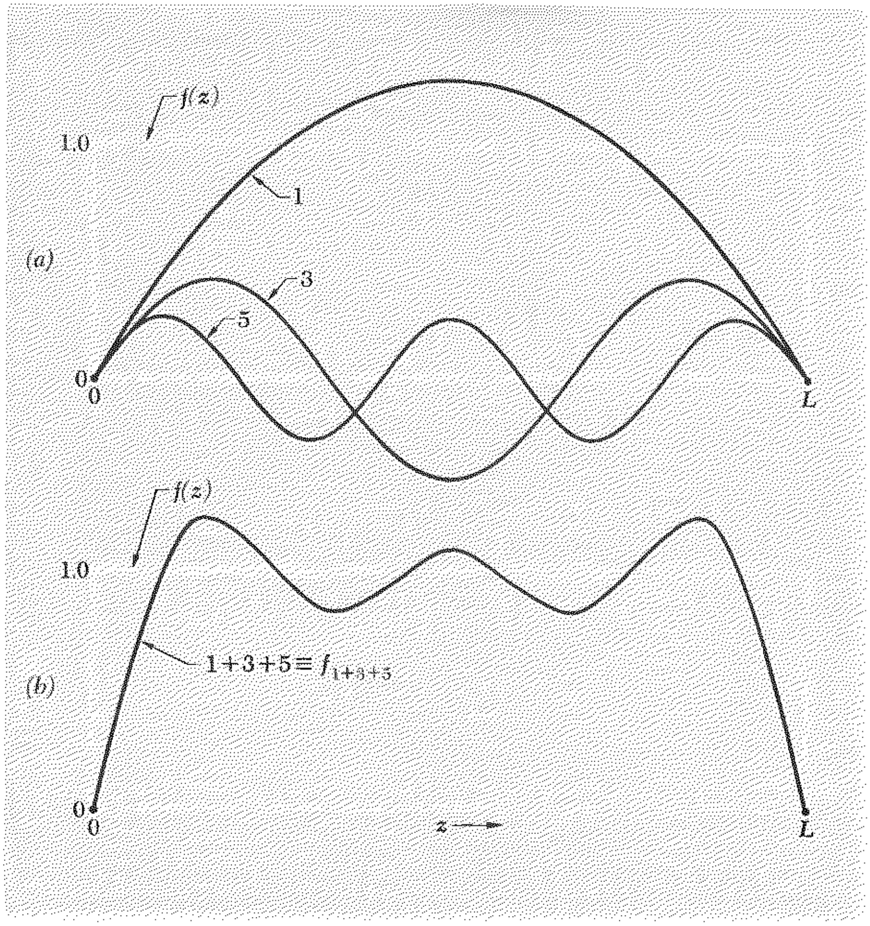

Fig. 2.7 Fourier analysis of square wave f(z). (a) Square wave f(z) and the first three contributions to its Fourier decomposition. The labels 1, 3, and 5 refer to the normal modes 1, 3, 5. (b) Square wave f(z) and the superposition f_{1+3+5} of its first three Fourier components.

In Fig. 2.7 are shown the square wave $f(z)$, the first three contributing terms given by Eq. (47), and the superposition of these first three terms.

Suppose that instead of trying to force a slinky into the configuration of the sharp-cornered function $f(z)$ which we have been considering, we constrain it at time zero to follow exactly the function

$$g(z) = 1.273 \sin \frac{\pi z}{L} + 0.424 \sin \frac{3\pi z}{L} + 0.255 \sin \frac{5\pi z}{L}. \tag{48}$$

This corresponds to the first three terms of Eq. (47) and is plotted in Fig. 2.7b. Now we let the slinky go at $t = 0$. What is $\psi(z,t)$? Does the shape remain constant as t increases? (See Prob. 2.16.)

Fourier analysis of a periodic function of time. Suppose we are given a function $F(t)$ that is defined for all t and that is periodic in t with period T_1:

$$F(t + T_1) = F(t) \qquad \text{for any } t. \tag{49}$$

We assume that $F(t)$ can be expanded in the Fourier series

$$F(t) = B_0 + \sum_{n=1}^{\infty} A_n \sin n\omega_1 t + \sum_{n=1}^{\infty} B_n \cos n\omega_1 t, \tag{50}$$

with

$$\omega_1 = 2\pi\nu_1 = \frac{2\pi}{T_1}. \tag{51}$$

The Fourier coefficients can be obtained directly from our results for the Fourier analysis of a spatially periodic function $F(z)$, which we studied above. The mathematical analysis cannot distinguish the variable $\theta = \omega_1 t$ from the variable $\theta = k_1 z$. Thus we obtain the results for the coefficients in Eqs. (50) directly from Eqs. (46):

$$\begin{aligned} B_0 &= \frac{1}{T_1}\int_{t_1}^{t_1+T_1} F(t)\,dt, \\ B_n &= \frac{2}{T_1}\int_{t_1}^{t_1+T_1} F(t)\cos n\omega_1 t\,dt, \\ A_n &= \frac{2}{T_1}\int_{t_1}^{t_1+T_1} F(t)\sin n\omega_1 t\,dt, \end{aligned} \tag{52}$$

where the time t_1 is any convenient time.

Sound of a piano chord. We shall illustrate this with a superposition of known ingredients, rather than by a Fourier analysis of a known function $F(t)$. Suppose you have a piano that is tuned to the "scientific scale." (See Home Exp. 2.6 if you want to know more about musical scales.) Let $\nu_1 = 128$ cps. That is the note C one octave (i.e., a factor of 2 in frequency) below middle C. Now let $\nu_3 = 3\nu_1 = 384$ cps. That is the G above middle C. Let $\nu_5 = 5\nu_1 = 640$ cps. That is the E above the G above middle C. Now strike all three notes at the same time. One hears a nice "open" chord. If you strike them at exactly the same time, and if you adjust your striking force so that the gauge pressure of air produced at your ear by the C128 string is (in appropriate units) $1.273 \sin 2\pi\nu_1 t$, pressure by the G384 string is $0.424 \sin 2\pi\nu_3 t$, and pressure by the E640 string is $0.255 \sin 2\pi\nu_5 t$, then the total air pressure $p(t)$ at your ear is the superposition

$$p(t) = 1.273 \sin 2\pi\nu_1 t + 0.424 \sin 2\pi\nu_3 t + 0.255 \sin 2\pi\nu_5 t. \tag{53}$$

But Eq. (53) is very similar to Eq. (48), which is plotted in Fig. 2.7*b*. All we have to do to obtain a plot of $p(t)$ is to change variables from $k_1 z$ to $\omega_1 t$

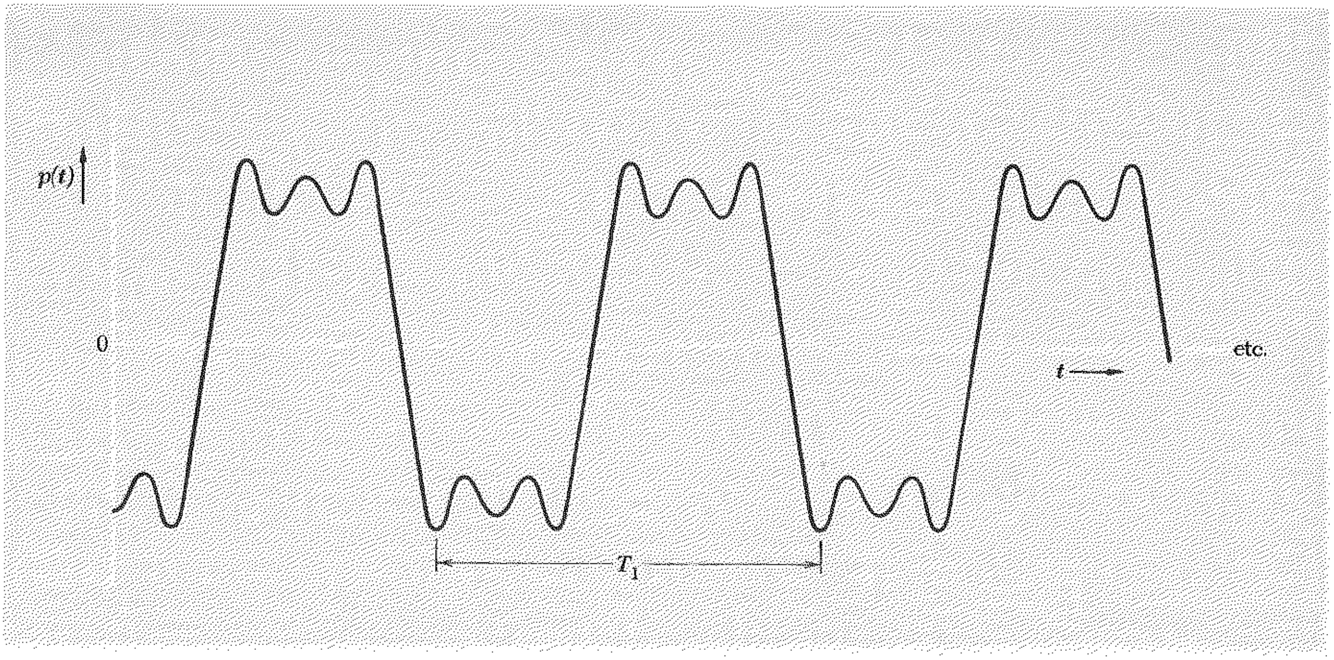

Fig. 2.8 Gauge pressure at ear due to superposition of the notes C128, G384, and E640 with the relative amplitudes and phases of Eq. (53). The period T_1 is (1/128) sec.

and extend the plot shown in Fig. 2.7*b*. Thus we get the result shown in Fig. 2.8.

If we do not strike all the keys at exactly the same time (i.e., to within an accuracy of much less than $\frac{1}{128}$ sec), the relative phases of the three notes will not be those of Eq. (53), and the superposition will not look like Fig. 2.8. But your ear does not notice this! Your ear (plus brain) performs a Fourier analysis on the total pressure. That must be so, because you "hear" the individual notes of the chord and recognize them. But the information as to relative phase of the notes is apparently discarded or perhaps not obtained. Otherwise you would notice a difference in the sound depending on the relative phases.

The pitch-detecting device in the ear is called the *basilar membrane.* It is enclosed in a fluid-filled, spiral-shaped organ in the inner ear called the *cochlea.* The cochlea is mechanically coupled to the eardrum. The end of the basilar membrane nearest the eardrum resonates at about 20,000 cps; the end farthest from the drum resonates at about 20 cps. Thus the extreme range of audible frequencies is about 20 cps to 20 kc. The cochlear nerve has sensors in the basilar membrane and "transduces" the mechanical vibrations into electrical signals that are carried to the brain, where they are somehow processed to become our hearing sensations. By doing the experiment of hitting the chord over and over and seeing that our sensation is the

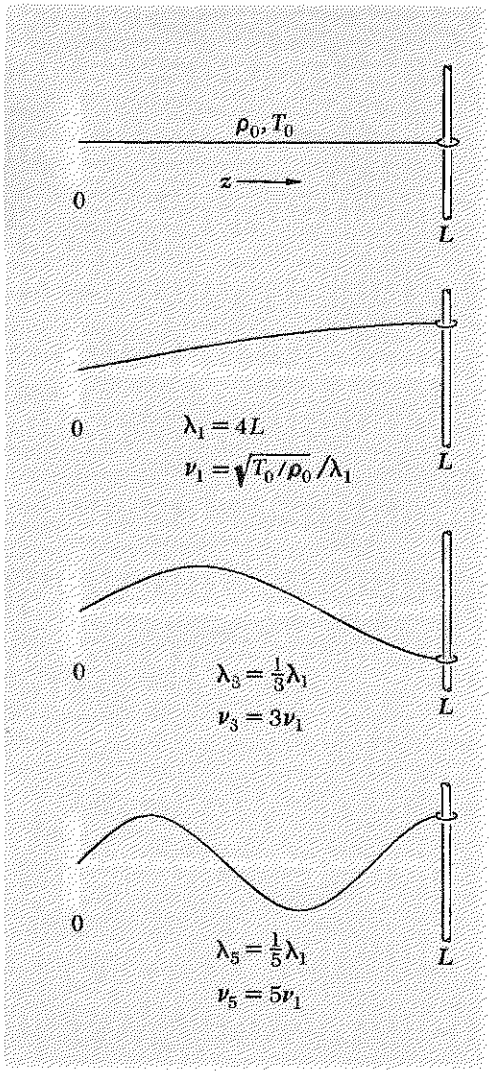

Fig. 2.9 Modes of continuous string with one fixed end and one free end.

same [even though $p(t)$ must have a very different shape depending on the relative phases], we have learned that somewhere the information as to the relative phase of the vibrations of different parts of the basilar membrane is lost. Perhaps this information is never picked up. Perhaps the transducer is a *square-law detector,* i.e., one that puts out an electrical signal proportional to the square of the amplitude of vibration of the membrane. Or perhaps the nerve signal does carry phase information [i.e., perhaps the signal does give $\psi(z,t)$ rather than $\psi^2(z,t)$], but the brain does not use the phase information, i.e., it does not form a superposition of $\psi(z,t)$ from different nerve signals. Apparently there is not much survival value in the phase information; otherwise in our evolutionary development we surely would have acquired some phase-detecting mechanism.

Other boundary conditions. In the general problem of transverse vibrations of a continuous string, it is not necessary that the string be fixed at both ends. One or both ends can be "free," at least as far as transverse oscillations are concerned. The tension and equilibrium configuration of the string can be maintained by a constraint in the form of a massless, frictionless ring sliding on a fixed rod oriented along x, i.e., transverse to the equilibrium axis of the string (which we always take along z). The normal modes will then have different configurations from those we obtained for the string with both ends fixed. The shapes of the modes are still sinusoidal functions of z, as given by Eq. (19). The dispersion relation between frequency and wavelength is still that given by Eq. (22). In fact our entire discussion preceding Eq. (23), the general solution for the displacement of the string in a single mode, is independent of the boundary conditions. It was only in the discussion following Eq. (23), that we specialized the solution to the case of the string fixed at $z = 0$ and L.

At a free end of a vibrating string, there is (by definition) no transverse force exerted on the end of the string, i.e., the frictionless rod exerts no transverse force on the frictionless ring. Then (by Newton's third law) the string and frictionless ring exert no transverse force on the frictionless rod. That means the string must be horizontal. *The slope of the string at a free end is zero at all times.* If one tries to exert a transverse force on the free end of a string, the string moves in such a way as to reduce the force to zero even as it is being applied. It never becomes different from zero, and the string remains horizontal, but of course not motionless. (The moral is that you cannot push on something that refuses to push back, but you can move it where you please.)

In Fig. 2.9 we show the modes of a string with one end fixed and the other free. We have labeled the successive modes according to the number of quarter-wavelengths contained in the string length L. Notice that the even harmonics with frequencies $2\nu_1$, $4\nu_1$, etc. are missing. The Fourier analysis of functions $f(z)$ with value zero at $z = 0$ and slope zero at $x = L$ is discussed in Prob. 2.29.

Dependence of tone quality on method of excitation. When a piano string is struck by its hammer, the fundamental (ν_1), the second harmonic or octave ($2\nu_1$), the octave plus a fifth ($3\nu_1$), the second octave ($4\nu_1$), the second octave plus a major third ($5\nu_1$), and the second octave plus a fifth ($6\nu_1$) are all excited to some extent, as are higher harmonics of the fundamental tone ν_1. The amount and phase of each Fourier component (each harmonic) depend on the initial configuration and velocity of all parts of the string at the instant just after it has been struck by the hammer. These depend to a great extent on the location of the hammer, i.e., on its distance from the end of the string. No mode that has a node (a permanently motionless point) at the striking point will be excited by the hammer blow, since the hammer imparts an initial velocity to the part of the string it hits. For example, if the string is plucked at its center, the modes with a node at the center are not excited. Inspection of Fig. 2.3 shows that in that case all the even harmonics are missing. Thus if we pluck the string for C128 in the middle, we expect it to vibrate in a superposition of C128, G384, E640, etc. The "tone quality" is then appreciably different from that produced when the string is struck near one end and vibrates in a superposition of C128, C256, G384, C512, E640, G768, etc.

Modes of homogeneous string form complete set of functions. Starting with a string fixed at both ends, we discovered that *any* reasonable function $f(z)$ that is defined between $z = 0$ and $z = L$ and that is zero at $z = 0$ and L can be expanded in the Fourier series

$$f(z) = \sum_{n=1}^{\infty} A_n \sin nk_1z; \qquad k_1L = \pi. \tag{54}$$

For that reason, the functions $\sin nk_1z$, with $n = 1, 2, 3, \ldots$, are said to be a *complete set* of functions [with respect to functions $f(z)$ that vanish at $z = 0$ and L]. A complete set of functions is defined as a set such that *any* (reasonable) function $f(z)$ can be written as a superposition of functions from the set by choosing suitable constant coefficients.

Inhomogeneous string. Besides the sinusoidal functions that constitute a Fourier series, are there other complete sets? Yes, infinitely many sets! We can see this as follows. Suppose that the string is not homogeneous, i.e., that either its mass density or its tension (or both) is a continuous function of position z. (An example of a "string" with varying density and tension is provided by a vertically hanging slinky with fixed top and bottom ends. The tension at the bottom is less than that at the top by the weight Mg, where M is the total mass of the slinky.) Then the equation of motion of a small segment of string does not again lead to the classical wave equation, which is

$$\frac{\partial^2\psi(z,t)}{\partial t^2} = \frac{T_0}{\rho_0}\frac{\partial^2\psi(z,t)}{\partial z^2}.$$

Instead, if we have equilibrium tension $T_0(z)$ and density $\rho_0(z)$, we easily find (Prob. 2.10) that we have

$$\frac{\partial^2\psi(z,t)}{\partial t^2} = \frac{1}{\rho_0(z)}\frac{\partial}{\partial z}\left[T_0(z)\frac{\partial\psi(z,t)}{\partial z}\right], \tag{55}$$

which reduces to the classical wave equation only if $T_0(z)$ and $\rho_0(z)$ are constants, independent of z. In a normal mode of this *inhomogeneous string*, just as in a mode of the homogeneous string, every part of the string vibrates in harmonic motion with the same frequency and phase constant:

$$\psi(z,t) = A(z)\cos(\omega t + \varphi). \tag{56}$$

Thus

$$\frac{\partial^2\psi}{\partial t^2} = -\omega^2 A(z)\cos(\omega t + \varphi), \tag{57}$$

$$\frac{\partial\psi}{\partial z} = \cos(\omega t + \varphi)\frac{dA(z)}{dz}. \tag{58}$$

Substituting these in Eq. (55) and canceling the common factor $\cos(\omega t + \varphi)$ yields the equation for the shape of the mode:

$$\frac{1}{\rho_0(z)}\frac{d}{dz}\left[T_0(z)\frac{dA(z)}{dz}\right] = -\omega^2 A(z). \tag{59}$$

Sinusoidal shape of standing waves is characteristic of homogeneous systems. The shape of the mode is given by $A(z)$, which is obtained by solving the differential equation Eq. (59) with the appropriate boundary conditions that $A(z) = 0$ at $z = 0$ and L. The function $A(z)$ is not sinusoidal in shape unless T_0 and ρ_0 are constants. Thus *sinusoidal* oscillations in space are only characteristic of the shapes of the normal modes of a *homogeneous* system.

Modes of inhomogeneous string form complete set of functions. We shall tell you without proof the characteristics of the normal modes for an inhomogeneous string with ends fixed at $z = 0$ and L. The lowest mode corresponds to a solution of Eq. (59), $A_1(z)$, which is zero only at $z = 0$ and L. (That is like one half-wavelength of a "distorted sine wave," which has no nodes between 0 and L.) This mode has frequency ω_1. The next mode has *one node* between $z = 0$ and L and thus resembles one full wavelength of a distorted sine wave. It has characteristic frequency ω_2. The mth mode has $m - 1$ nodes between $z = 0$ and L and resembles m half-wavelengths of a distorted sine wave. There are an infinite number of modes (for a continuous string). The functions $A_1(z)$, $A_2(z)$, $A_3(z)$, . . . , which give the space dependence of the modes, form a complete set with respect to any reasonable function $f(z)$ that vanishes at $z = 0$ and L. A reasonable function $f(z)$ is defined to be one which the string or slinky can follow without violating any of our assumptions. In that case we can make a template that has the shape $f(z)$, fit the inhomogeneous string to the template, and let it go from rest at $t = 0$. The string will vibrate in an infinite superposition of its modes:

$$\psi(z,t) = \sum_{m=1}^{\infty} c_m A_m(z) \cos \omega_m t \tag{60}$$

Then at $t = 0$ we have

$$\psi(z,0) = f(z) = \sum_{m=1}^{\infty} c_m A_m(z). \tag{61}$$

Equation (61) shows that $f(z)$ (subject to our assumptions) can be expanded in the set of functions $A_m(z)$. Thus $A_m(z)$ form a complete set of functions. This argument is exactly analogous to the one that convinced us that the sinusoidal functions of a Fourier series form a complete set with respect to functions $f(z)$ that vanish at $z = 0$ and L.

Eigenfunctions. There are an infinite number of different ways that we can construct a string with nonuniform mass density and tension. Therefore, there are an infinite number of different complete sets $A_m(z)$. Sinusoidal functions of z are thus not the only complete set of functions for expanding functions $f(z)$. But they are a very important set, because they are very simple and easy to understand. Furthermore, they give the shapes of the modes whenever we have a system that is spatially *homogeneous*. When that is not the case, the sinusoidal functions are not very useful. Instead one tries to find and use the appropriate functions $A_m(z)$ that correspond to the normal modes of the system. These functions $A_m(z)$, or, more generally, $A_m(x,y,z)$ for a three-dimensional system, are called *eigenfunctions*. They give the *space* dependence of the normal modes.

For every position x, y, z, the *time* dependence of a mode is *always* given by $\cos(\omega t + \varphi)$. Thus a mode is essentially nothing but the simultaneous small oscillation (small enough to give linear equations) of all the moving parts, all parts oscillating with the same frequency and same phase constant. When the entire system is in a single mode, it pulsates and throbs like one big oscillator. Each mode has its own "shape," i.e., its own eigenfunction. The relation between mode frequency and shape is called the dispersion relation, $\omega(k)$, when the shapes of the eigenfunctions are sinusoidal. When they are not sinusoidal, there is, of course, no such thing as wavelength or wavenumber k. Then the relation between mode frequency and shape is not usually called by the name "dispersion relation."