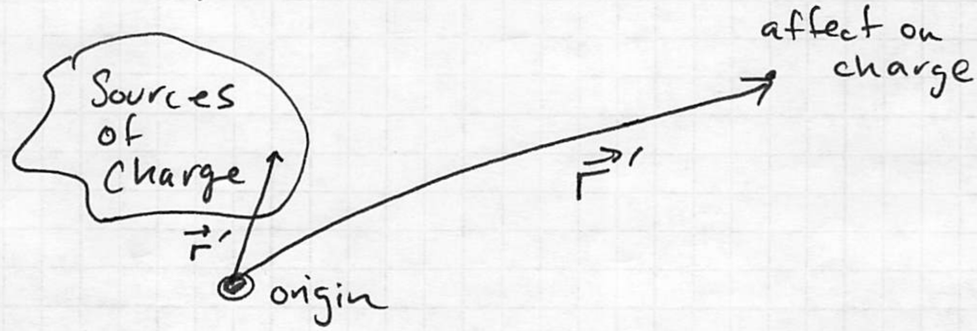
Now, we will begin our study of electrostatics. In this part of the class, we will be uniquely concerned with the problem described below:



In 184, you spent a fair amount of time concerned with the forces between charges (q_1 & q_2)

$$F = \frac{k q_1 q_2}{r^2}$$; here the force magnitude is shown.

In this class, our thinking will begin with force, $\vec{F}$, but we will shift to the concept of electric field, $\vec{E}$, quite quickly as it doesn't require the concept of a test charge.

<u>Superposition</u> is <u>incredibly</u> <u>powerful</u>!

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots = \sum \vec{F}_i$$

the net force on an object is the result of the vector sum of all individual contributions.

We have a quantity called electric field $\vec{E}$, which is the force per unit charge

$$\vec{E} = \vec{F}/Q_{test}$$

If this force acts on a test charge due to a bunch of sources then,

$$\vec{F}_{net} = Q\vec{E}_1 + Q\vec{E}_2 + Q\vec{E}_3 + \cdots = \sum Q\vec{E}_i$$
$$= Q\sum \vec{E}_i = Q\vec{E}_{net}$$

that is, there is a net electric field due to all charges that are not Q, which give rise to the net force on Q.

<u>the Electric Field</u> obeys <u>superposition</u>!

This is the <u>crux of the biscuit</u>! It is incredibly powerful

take any distribution of charge
& just add up its effects

$\sum$  $\vec{E}_i$'s $\Longrightarrow \vec{E}_{net}$

How do we do that? Treat small chunks of charge like a point charge.

$$\vec{E}_i = \frac{1}{4\pi\epsilon_0} \frac{q_i}{|\vec{r}-\vec{r}_i{}'|^3}(\vec{r}-\vec{r}_i{}') = \frac{1}{4\pi\epsilon_0}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i \Rightarrow \sum_i \vec{E}_i$$

where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

☆ Clicker Questions: 5 and 4 charges.

What happens if the charges are smeared out? Superposition still works!

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{\vec{\mathscr{r}}_i^{\,2}}\hat{\mathscr{r}}_i \longrightarrow \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}^2}\hat{\mathscr{r}}$$

we integrate over the distribution!

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}^2}\hat{\mathscr{r}}$$

$\vec{\mathscr{r}}$ now points from dq to the observation location.

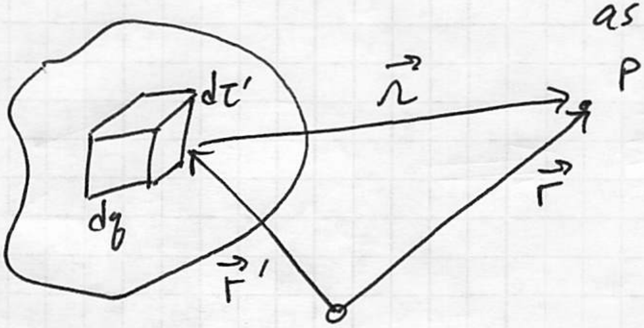
We will encounter different kinds of distributions throughout our work.

* Let's start with charge distributed throughout a volume: charge density: $\rho(\vec{r}\,') \equiv$ charge/volume

We say the charge contained by the infinitesimal volume $d\tau'$ (Griffith's usuage) is,

$$dq = \rho\, d\tau'$$

you may have seen $d\tau'$ as d^3r' or $(dx'dy'dz')$



As usual $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

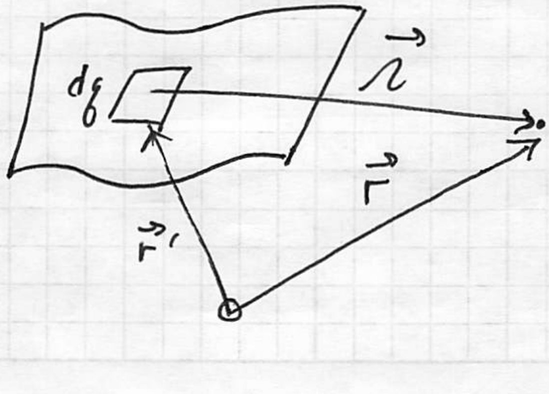
Summing over dq's means $\int \rho(\vec{r}\,')\,d\tau' \ldots$

so,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\iiint_{\text{volume}} \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}^2}\hat{\mathscr{r}}$$

with $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

* If charges live on a flat (or 2-D) surface, use $\sigma(\vec{r}\,') \equiv$ charge/area so $dq = \sigma\, dA'$

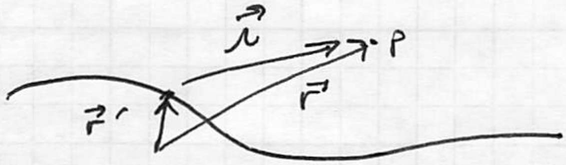


$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\iint_{\text{Area}} \frac{\sigma(\vec{r}\,')}{\mathscr{r}^2}\,dA\,\hat{\mathscr{r}}$$

If charges all live on a line (1-D), use

$\lambda \equiv$ charge/length

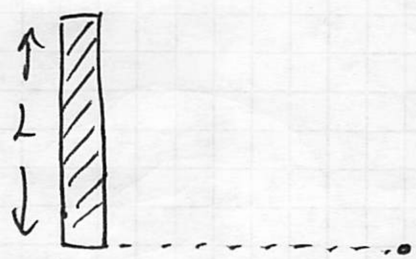
✱ Activity:
Draw your own picture,
What's $\vec{E}$?



$dq = \lambda\, dl'$

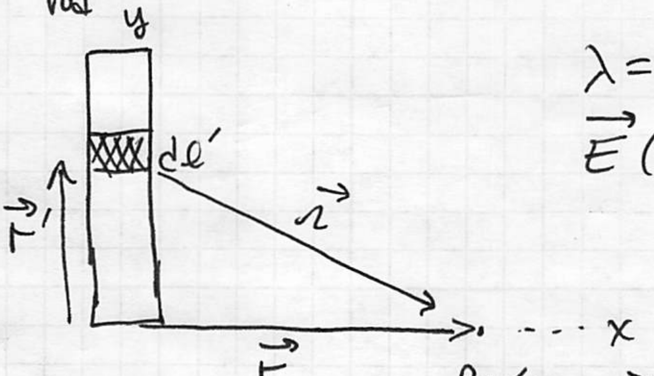
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{line}} \frac{\lambda(\vec{r}')\, dl'}{\mathscr{r}^2}\, \hat{\mathscr{r}}$$

Example: Like Griffiths 2.1 with a twist.



Let's find $\vec{E}$ at the point P shown.

Let's apply a coordinate system to the problem and set this up.



$\lambda = Q/L$ is given

$\vec{E}(x,0,0) = ?$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{line}} \frac{\lambda\, dl'}{\mathscr{r}^2}\, \hat{\mathscr{r}}$$

But that's just the formula!
We need to compute this integral!

As posed, there's no symmetry to exploit in this problem (like in Griffiths 2.1), there will be both nonzero E_x & E_y! Arggh!!

We will do E_x together; E_y is left as an exceerise for you.
(ahh! spelling)

To get started, we need to setup the integral,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda\, dl'}{\mathscr{r}^2}\,\hat{\mathscr{r}}$$

★ Clicker Question: what is $|\vec{\mathscr{r}}|$?

Another way to write $\vec{E}(\vec{r})$ is this:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda\, dl'}{\mathscr{r}^3}\,\vec{\mathscr{r}}$$

← notice: $\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{|\vec{\mathscr{r}}|}$

★ Clicker Question: We can write

$$E_x(x,0,0) = \frac{\lambda}{4\pi\epsilon_0}\int \ldots$$

; what is ....?

We can now put this all together:

$$E_x(x,0,0) = \frac{1}{4\pi\epsilon_0}\int_{y'=0}^{y'=L} \frac{\lambda\, dy'}{x^2+y'^2}\,\frac{x}{\sqrt{x^2+y'^2}}$$

Yuck! How do we deal with this?

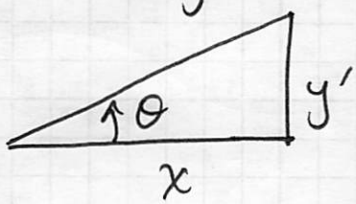
1. Look it up!
2. Use python (or something else)
3. Do it!

$$E_x(x,0,0) = \frac{1}{4\pi\epsilon_0}\int_0^L \frac{\lambda x\, dy'}{(x^2+y'^2)^{3/2}}$$

Let's do this integral,

this is one of those "trig" integrals,

they are very common in E+M.



$$y' = x\tan\theta$$

$$dy' = \frac{x}{\cos^2\theta}\,d\theta$$

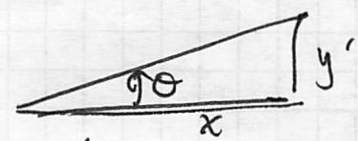
$$(x^2+y'^2)^{3/2} = x^3\left(1 + y'^2/x^2\right)^{3/2}$$

$$= x^3(1+\tan^2\theta)^{3/2} = x^3\left(\frac{1}{\cos^2\theta}\right)^{3/2} = \frac{x^3}{\cos^3\theta}$$

$$E_x(x,0,0) = \frac{\lambda}{4\pi\epsilon_0}\int_0^L \frac{x\,dy'}{(x^2+y^2)^{3/2}} \quad \text{becomes}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\int \frac{x\,(x/\cos^2\theta\,d\theta)}{x^3\,(1/\cos^3\theta)} = \frac{\lambda}{4\pi\epsilon_0}\,\frac{1}{x}\int \cos\theta\,d\theta$$

$$E_x = \frac{\lambda}{4\pi\epsilon_0 x}\Big[\sin\theta\Big]\Big|_{y'=0}^{y'=L}$$



$$\frac{y'}{\sqrt{x^2+y'^2}}$$

$$E_x = \frac{\lambda}{4\pi\epsilon_0 x}\,\frac{y'}{(x^2+y'^2)^{1/2}}\Bigg|_0^L$$

total Q!

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\,\frac{1}{x}\,\frac{L}{(x^2+L^2)^{1/2}} = \frac{\lambda L}{4\pi\epsilon_0}\,\frac{1}{x}\,\frac{1}{(x^2+L^2)^{1/2}}$$

We have generated a new expression - one that you are likely to not have seen before. How do we check if its a reasonable solution? (Ask class)

① <u>Check it's units</u>

$[E_x] = [N/c]$ newtons/coulomb is what we should get.

$\left[\frac{1}{4\pi\epsilon_0}\right] = \left[\frac{Nm^2}{C^2}\right]$ $\quad [\lambda] = [C/m]$

$$[E_x] = \left[\frac{1}{4\pi\epsilon_0}\right][\lambda][L]\left[\frac{1}{x}\right]\left[\frac{1}{(x^2+L^2)^{1/2}}\right]$$

$$= \left[\frac{N\cancel{m^2}}{C\cancel{^2}}\right]\left[\frac{\cancel{C}}{\cancel{m}}\right][\cancel{m}]\left[\frac{1}{\cancel{m}}\right]\left[\frac{1}{\cancel{m}}\right]$$

$$= \left[\frac{N}{C}\right] \checkmark \quad \text{units are ok!}$$

② <u>Limiting Behavior</u>

CQ: What happens as you get really far from the rod?

<u>One way</u>: $x \to \infty$

$$E_x = \frac{\lambda}{4\pi\epsilon_0} \frac{L}{x\sqrt{x^2+L^2}}$$ as x gets really big,

$$1/x \to 0$$

thus $$E_x \to 0 \quad \text{as } x \to \infty.$$

this misses key information:

How does it go to zero? What function could tell you how it goes to zero?

To do this, we will use a <u>Taylor Expansions</u>

<u>Second Way</u>: Approximate using a Taylor Series

<u>Formally</u>, expand a function around x_0

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0) + \ldots$$

x_0: expansion point

* our series will converge "quickly" when x is close to x_0

Often, $x_0 = 0$ in our problems we end up doing a MacLaurin Series,

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2 + \ldots$$

* again, this converges "quickly" if x is small, i.e., x is close to zero.

Back to E_x,

so if $x \to \infty$ $\quad L/x \ll 1$ that is L/x is close to zero so let's expand E_x around $L/x = 0$

<u>Activity</u>: Expand E_x around $\frac{L}{x} = 0$,

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\frac{L}{x\sqrt{x^2+L^2}} = \frac{\lambda}{4\pi\epsilon_0}\frac{L}{x^2\sqrt{1+L^2/x^2}}$$

To do this, expand

$$f\left(\frac{L}{x}\right) = \frac{1}{\sqrt{1+L^2/x^2}} = f(0) + f'(0)\left(\frac{L}{x}\right) + \frac{1}{2}f''(0)\left(\frac{L}{x}\right)^2 + \ldots$$

$$\frac{1}{\sqrt{1+L^2/x^2}} \simeq 1 - \frac{1}{2}\left(\frac{L}{x}\right)^2 + \ldots$$

The derivative (formal) method is a pain!
Typically, we don't need more than first (or second) order,

Good Rule (b/c its a common expansion):

$$\frac{1}{(1+x)^n} \simeq 1 - nx \qquad \frac{1}{(1-x)^n} \simeq 1 + nx$$

so,

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{x^2}\frac{L}{\sqrt{1+L^2/x^2}} \simeq \frac{\lambda}{4\pi\epsilon_0}\frac{L}{x^2}\left(1 - \frac{1}{2}\frac{L^2}{x^2}\right)$$

$\lambda L = Q$ so,

$$E_x = \underbrace{\frac{Q}{4\pi\epsilon_0}\frac{1}{x^2}}_{\text{point charge}}\left(1 - \underbrace{\frac{1}{2}\frac{L^2}{x^2}}_{\text{small correction}}\right)$$

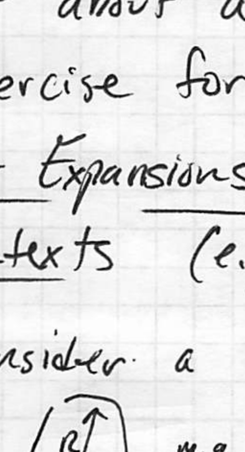
point charge + small correction

so, E_x looks like pt. charge far away.

what about as $x \to 0$ (i.e. $x/L \ll 1$)?
(Exercise for you.)

Taylor Expansions can be used in a wide variety of contexts (e.g. Differential Equations)

Consider a charge (m, q) near a disk (σ, R)



Suppose you want model the motion the charge, q, near the disk.

$$E_{Disk} = \frac{2\pi\sigma}{4\pi\epsilon_0}\left[1 - \frac{x}{(x^2+R^2)^{1/2}}\right]$$

<u>model of motion</u>

$$m\ddot{x} = F_{elec} = qE_{disk}$$

$$\ddot{x} = \underbrace{\frac{q}{m}\frac{2\pi\sigma}{4\pi\epsilon_0}}\left[1 - \frac{x}{(x^2+R^2)^{1/2}}\right]$$

these are all constants, call them "C".

$$\ddot{x} = C\left[1 - \frac{x}{(x^2+R^2)^{1/2}}\right]$$ we get a

Non linear Differential Equation !!!

what if the disk is very large (electron and "capacitor" plate) or charge is very close?

$x/R \ll 1$ so,

$$\cancel{\ddot{x} = C\left[1 - \frac{x}{R}\left(\frac{1}{(x/R^2 + R^2)^{1/2}}\right)\right]}$$

$$\ddot{x} = C\left[1 - \frac{x}{R}\frac{1}{(1+x^2/R^2)^{1/2}}\right]$$

$$= C\left[1 - \frac{x}{R}\left(1 - \frac{1}{2}\frac{x^2}{R^2}\right)\right]$$

$$= C\left[1 - \frac{x}{R} + \frac{1}{2}\frac{x^3}{R^3}\right]$$

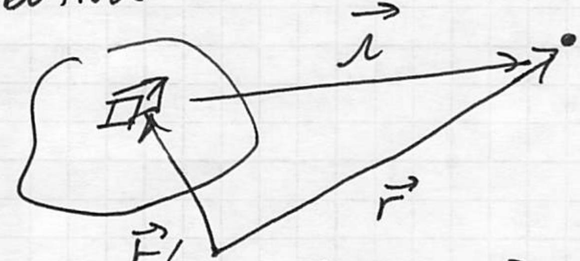
$\hookrightarrow 0$ b/c third order

so,

$$\ddot{x} = \frac{q}{m}\frac{2\pi\sigma}{4\pi\epsilon_0}\left[1 - \frac{x}{R}\right]$$ linear differential equation!

check? units! careful: solution only valid when $x/R \ll 1$

You have seen how we can solve for the electric field for a continuous charge distribution



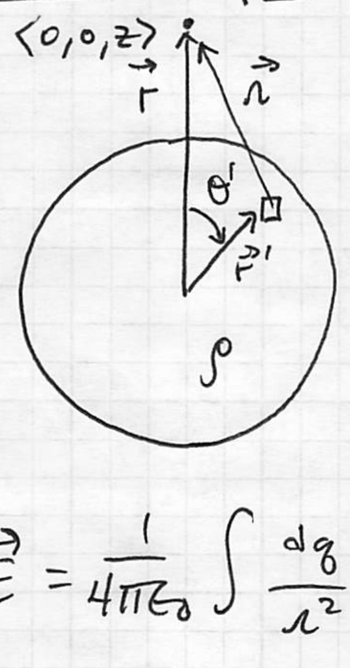
add all the contributions $\sum \vec{E}_i \Rightarrow \int d\vec{E} = \vec{E}(\vec{r})$ obtain net field.

<u>Activity</u>: What about if the distribution is not-integrable (i.e., there's no anti-derivative)?

How do we use a computer to do this?

<u>Activity</u>: Work together to come up w/ steps needed to solve this problem computationally

<u>One last example</u>: Find the electric field at a point P directly above a uniformly charged sphere, ρ. By symmetry we are solving for any pt. a distance z from <u>center</u>. If $\vec{r} = \langle 0, 0, z \rangle$, $E_x = E_y = 0$ by symmetry (convince yourself!)



$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{\mathscr{r}^2} \hat{\mathscr{r}}$$

$dq = \rho\, d\tau' = \rho\, dx'\, dy'\, dz'$

Better use spherical coords.

$dq = \rho r'^2 \sin\theta\, dr'\, d\theta'\, d\phi'$

CQ!
what is $|\vec{\mathscr{r}}|$?

$\vec{r} = z\hat{z}$ $\quad$ $\vec{r}'$ is shown

$$|\vec{\mathscr{r}}| = \sqrt{z^2 + r'^2 - 2zr'\cos\theta}$$ Law of cosines!

$$\hat{\mathscr{r}}_z = \frac{\vec{\mathscr{r}}_z}{|\vec{\mathscr{r}}|} = \frac{z - r'\cos\theta'}{\sqrt{z^2 + r'^2 - 2zr'\cos\theta'}}$$ (Convince yourself of $\vec{\mathscr{r}}_z$!)

the integral is hairy looking but totally doable.
But there's a better way...