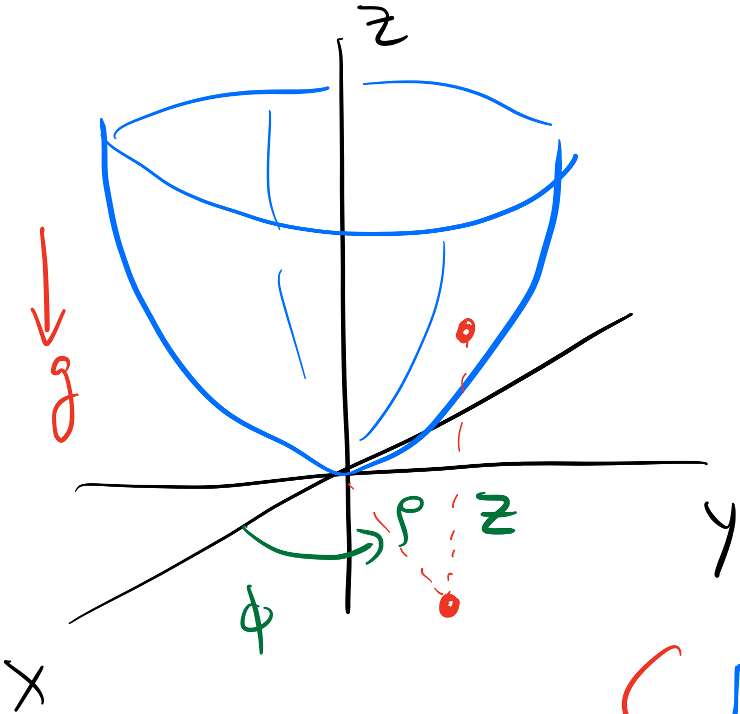
Bead in a paraboloid



cylindrical coordinates

$\langle \rho, \phi, z \rangle$

$$v^2 = \dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2$$

$$z = c\rho^2$$

Important for sensemaking

Note c has units

$[z] = m$

$[\rho^2] = m^2 \quad \therefore \quad [c] = \frac{1}{m}$

$T = \frac{1}{2} m v^2 \Rightarrow$ in cylindrical

$$T = \frac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2 \right)$$

pt. particle kinetic energy

$U = mgz$ grav potential energy (pt. particle near Earth)

$$\mathcal{L} = \frac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2 \right) - mgz$$

$$\mathcal{L}(\rho, \dot\rho, \phi, \dot\phi, z, \dot z, t)$$

Constraint

$z = c\rho^2$

$\dot z = 2c\rho\dot\rho$

$$\Downarrow$$

$$\mathcal{L}(\rho, \dot\rho, \phi, \dot\phi, t) \quad \text{but really} \quad \mathcal{L}(\rho, \dot\rho, \dot\phi)$$

no ϕ dependence $\Rightarrow$ expect L conservation

no explicit t dependence $\Rightarrow$ expect Energy conservation

$$\mathcal{L} = \frac{1}{2}m\left(\dot\rho^2 + \rho^2\dot\phi^2 + (2c\rho\dot\rho)^2\right) - mgc\rho^2$$

$$= \frac{1}{2}m\left(\dot\rho^2 + \rho^2\dot\phi^2 + 4c^2\rho^2\dot\rho^2\right) - mgc\rho^2$$

EOM for ϕ why? b/c only $\dot\phi^2$ term

$$\underbrace{\frac{\partial\mathcal{L}}{\partial\phi}}_{0} - \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot\phi}\right) = 0$$

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot\phi}\right) = \frac{d}{dt}\left(\underbrace{m\rho^2\dot\phi}_{L_z}\right) = 0$$

z component of ang. mom is conserved!

EOM for ρ

$$\mathcal{L} = \frac{1}{2}m\left(\dot\rho^2 + \rho^2\dot\phi^2 + 4c^2\rho^2\dot\rho^2\right) - mgc\rho^2$$

$$\frac{\partial \mathcal{L}}{\partial \rho} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\rho}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \rho} = m\rho\dot\phi^2 + 4c^2\rho\dot\rho^2 m - 2mgc\rho$$

$$\frac{\partial \mathcal{L}}{\partial \dot\rho} = m\dot\rho + 4mc^2\rho^2\dot\rho$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\rho}\right) = m\ddot\rho + 4mc^2\left(2\rho\dot\rho^2 + \rho^2\ddot\rho\right)$$

So,

$$\cancel{m}\ddot\rho + 4\cancel{m}c^2\left(2\rho\dot\rho^2 + \rho^2\ddot\rho\right) = \cancel{m}\rho\dot\phi^2 + 4c^2\rho\dot\rho^2\cancel{m} - 2\cancel{m}gc\rho$$

$$\ddot\rho\left(1 + 4c^2\rho^2\right) + 8c^2\rho\dot\rho^2 = \rho\dot\phi^2 + 4c^2\rho\dot\rho^2 - 2gc\rho$$

$$\boxed{\ddot\rho\left(1 + 4c^2\rho^2\right) + 4c^2\rho\dot\rho^2 - \rho\dot\phi^2 + 2gc\rho = 0}$$

also $\boxed{\frac{d}{dt}\left(m\rho^2\dot\phi\right) = 0}$ or $2\rho\dot\rho\dot\phi + \rho^2\ddot\phi = 0$

Clean up into $\ddot{\rho} = f(\rho, \dot{\rho}, \dot{\phi}, t)$

$\ddot{\phi} = g(\dot{\phi}, t)$

$$\boxed{\ddot{\rho} = \frac{\rho\dot{\phi}^2 - 4c^2\rho\dot{\rho}^2 - 2cg\rho}{(1 + 4c^2\rho^2)}}$$

$$\ddot{\phi} = -\frac{2\rho\dot{\rho}\dot{\phi}}{\rho^2}$$

$\Rightarrow$ solve away from $\langle 0,0,0 \rangle$

$\Downarrow$

$$\boxed{\ddot{\phi} = -\frac{2\dot{\rho}\dot{\phi}}{\rho}}$$

Prepare for Numerical Integration

Let $\omega = \dot{\phi}$ and $v = \dot{\rho}$ then,

$$\dot{v} = \frac{\rho\omega^2 - 4c^2\rho v^2 - 2cg\rho}{(1 + 4c^2\rho^2)} \qquad \dot{\omega} = -\frac{2\dot{\rho}\dot{\phi}}{\rho}$$

$$\dot{\rho} = v \qquad \dot{\phi} = \omega$$

4 1st order ODEs to solve