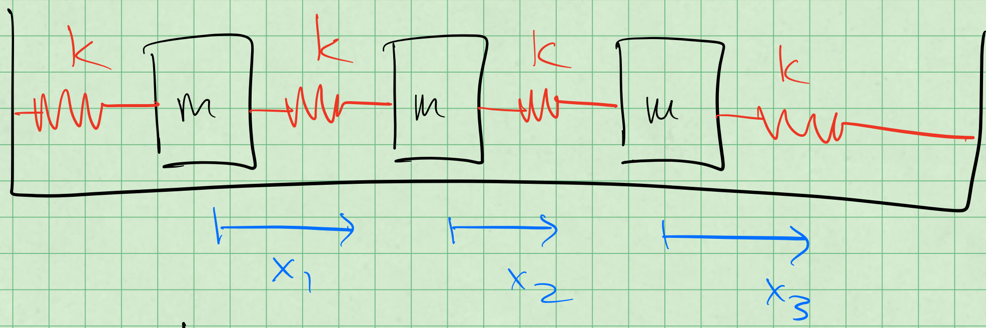


$$T = \frac{1}{2}k\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right)$$

$$U = \frac{1}{2}k x_1^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}(x_3 - x_2)^2 + \frac{1}{2}k x_3^2$$

$$\mathcal{L} = T - U$$

$$= \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) - \frac{1}{2}k\left(x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + x_3^2\right)$$

$$\frac{d\mathcal{L}}{dx_i} - \frac{d}{dt}\frac{d\mathcal{L}}{d\dot{x}_i} = 0 \Rightarrow \underline{\text{EoMs}}$$

Equations of Motion

x_1: $-kx_1 + k(x_2 - x_1) - m\ddot{x}_1 = 0$

x_2: $-k(x_2 - x_1) + k(x_3 - x_2) - m\ddot{x}_2 = 0$

x_3: $-k(x_3 - x_2) - kx_3 - m\ddot{x}_3 = 0$

Change to Matrix representation

$$\ddot{\vec{x}} = \underline{\underline{A}}\,\vec{x} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \ddot{\vec{x}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}$$

$$\ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2 \boxed{+0x_3}$$

$$\ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2 + \frac{k}{m}x_3$$

$$\ddot{x}_3 = \boxed{0x+} \frac{k}{m}x_2 - \frac{2k}{m}x_3$$

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & +\frac{k}{m} & 0 \\ \frac{k}{m} & -\frac{2k}{m} & \frac{k}{m} \\ 0 & \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Now we seek normal mode solutions

$x \sim e^{i\omega t}$, this gives the following

eigen value problem

$$\ddot{\vec{x}} = \underline{\underline{A}}\,\vec{x} = -\omega^2\vec{x}$$

diffy Q w/ assumed solution

let $\lambda = -\omega^2$

$\underline{\underline{A}}\,\vec{x} = \lambda\vec{x}$ $\qquad$ $\det\left|\underline{\underline{A}} - \underline{\underline{I}}\lambda\right| = 0$

gives eigenvalues

$$\det\left|\underline{\underline{A}} - \underline{\underline{I}}\lambda\right| = 0$$

$$= \left(-\frac{2k}{m}-\lambda\right)\left[\left(-\frac{2k}{m}-\lambda\right)^2 - \frac{k^2}{m^2}\right] - \frac{k}{m}\left[\left(\frac{k}{m}\right)\left(-\frac{2k}{m}-\lambda\right)\right] = 0$$

$$= \left(-\frac{2k}{m}-\lambda\right)\left[\left(-\frac{2k}{m}-\lambda\right)^2 - \frac{2k^2}{m^2}\right] = 0$$

3 solutions

$$\left(-\frac{2k}{m}-\lambda\right) = 0 \Rightarrow \boxed{\lambda = -\frac{2k}{m}}$$

$$\left(-\frac{2k}{m}-\lambda\right)^2 - \frac{2k^2}{m^2} = 0 \quad \left(-\frac{2k}{m}-\lambda\right)^2 = \frac{2k^2}{m^2}$$

$$\left(-\frac{2k}{m}-\lambda\right) = \pm\sqrt{2}\,\frac{k}{m}$$

$$\lambda = \left(-2 \mp \sqrt{2}\right)\frac{k}{m}$$

Note all λ's < 0 b/c $\lambda \equiv -\omega^2$

so,

$$\boxed{\begin{aligned} \omega_1^2 &= \frac{2k}{m} \\ \omega_2^2 &= \left(2-\sqrt{2}\right)\frac{k}{m} \\ \omega_3^2 &= \left(2+\sqrt{2}\right)\frac{k}{m} \end{aligned}}$$

Cool. but what about the nature of the modes?

Let's look at one,

$$\omega_1 = \sqrt{\frac{2k}{m}}$$

$\underline{\underline{A}}\,\vec{x}_1 = -\omega_1^2 \vec{x}_1$ "$\vec{x}_1$ is the eigenfunction solving the eigenvalue problem with ω_1"

(1) $-\frac{2k}{m}x_1 + \frac{k}{m}x_2 = -\omega_1^2 x_1 = -\frac{2k}{m}x_1$

(2) $\frac{k}{m}x_1 - \frac{2k}{m}x_2 + \frac{k}{m}x_3 = -\omega_1^2 x_2 = -\frac{2k}{m}x_2$

(3) $\frac{k}{m}x_2 - \frac{2k}{m}x_3 = -\omega_1^2 x_3 = -\frac{2k}{m}x_3$

Note $\omega_1^2 = \frac{2k}{m}$, so....

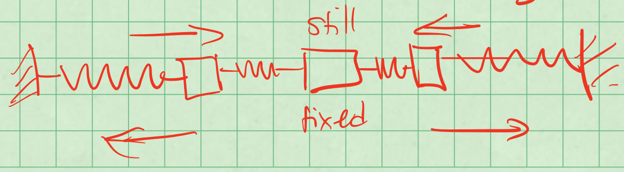
(1) $\Rightarrow$ $x_2 = 0$

(2) $\Rightarrow$ $x_1 + x_3 = 0$ $x_1 = -x_3$

(3) $\Rightarrow$ $x_2 = 0$ also!

So

$x_1 = 1$ $x_2 = 0$ $x_3 = -1$



opp motion