1.4 Free Oscillations of Systems with Two Degrees of Freedom

In nature there are many fascinating examples of systems having two degrees of freedom. The most beautiful examples involve molecules and elementary particles (the neutral K mesons especially); to study them requires quantum mechanics. Some simpler examples are a double pendulum (one pendulum attached to the ceiling, the second attached to the bob of the first); two pendulums coupled by a spring; a string with two beads; and two coupled LC circuits. (See Fig. 1.6.) It takes two variables to describe the configuration of such a system, say ψ_a and ψ_b. For example, in the case of a simple pendulum free to swing in any direction, the "moving parts" ψ_a and ψ_b would be the positions of the pendulum in the two perpendicular horizontal directions; in the case of coupled pendulums, the moving parts ψ_a and ψ_b would be the positions of the pendulums; in the case of two coupled LC circuits, the "moving parts" ψ_a and ψ_b would be the charges on the two capacitors or the currents in the circuits.

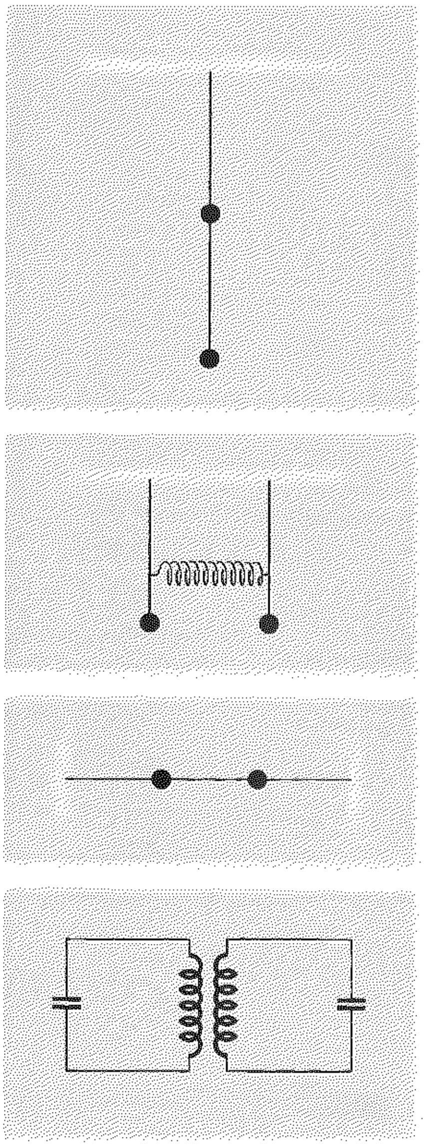


Fig. 1.6 Systems with two degrees of freedom. (The masses are constrained to remain in the plane of the figure.)

The general motion of a system with two degrees of freedom can have a very complicated appearance; no part moves with simple harmonic motion. However, we will show that for two degrees of freedom and for linear equations of motion the most general motion is a *superposition* of two independent simple harmonic motions, both going on simultaneously. These two simple harmonic motions (described below) are called *normal modes* or simply *modes*. By suitable starting conditions (suitable initial values of ψ_a, ψ_b, $d\psi_a/dt$, and $d\psi_b/dt$), we can get the system to oscillate in only one mode or the other. Thus the modes are "uncoupled," even though the moving parts are not.

Properties of a mode. When only one mode is present, each moving part undergoes simple harmonic motion. All parts oscillate with the same frequency. All parts pass through their equilibrium positions (where ψ is zero) simultaneously. Thus, for example, one never has in a single mode, $\psi_a(t) = A\cos\omega t$ and $\psi_b(t) = B\sin\omega t$ (different phase constants) or $\psi_a(t) = A\cos\omega_1 t$ and $\psi_b(t) = B\cos\omega_2 t$ (different frequencies). Instead one has, for one mode (which we call mode 1),

$$\begin{aligned}\psi_a(t) &= A_1\cos(\omega_1 t + \varphi_1),\\ \psi_b(t) &= B_1\cos(\omega_1 t + \varphi_1) = \frac{B_1}{A_1}\psi_a(t),\end{aligned} \tag{41}$$

with the *same frequency and phase constant* for both degrees of freedom (moving parts). Similarly, for mode 2, the two degrees of freedom a and b move according to

$$\begin{aligned}\psi_a(t) &= A_2\cos(\omega_2 t + \varphi_2),\\ \psi_b(t) &= B_2\cos(\omega_2 t + \varphi_2) = \frac{B_2}{A_2}\psi_a(t).\end{aligned} \tag{42}$$

Each mode has its own characteristic frequency: ω_1 for mode 1, ω_2 for mode 2. In each mode the system also has a characteristic "configuration" or "shape," given by the ratio of the amplitudes of motion of the moving parts: A_1/B_1 for mode 1 and A_2/B_2 for mode 2. Note that in a mode the ratio $\psi_a(t)/\psi_b(t)$ is constant, independent of time. It is given by the appropriate ratio A_1/B_1 or A_2/B_2, which can be either positive or negative.

The most general motion of the system is (as we will show) simply a superposition with both modes oscillating at once:

$$\begin{aligned}\psi_a(t) &= A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \varphi_2),\\ \psi_b(t) &= B_1 \cos(\omega_1 t + \varphi_1) + B_2 \cos(\omega_2 t + \varphi_2).\end{aligned} \tag{43}$$

Let us consider some specific examples.

Example 6: Simple spherical pendulum

This example is almost too simple, for it does not reveal the full richness of complexity of the general motion that corresponds to Eqs. (43) because the two modes, corresponding respectively to oscillation in the x and in the y direction, have the same frequency, given by $\omega^2 = g/l$. Rather than the superpositions of Eq. (43), corresponding to two different frequencies, we have the simpler results obtained in Eqs. (39) and (40)

$$\begin{aligned}x(t) \equiv \psi_a(t) &= A_1 \cos(\omega_1 t + \varphi_1), \qquad \omega_1 = \omega,\\ y(t) \equiv \psi_b(t) &= B_2 \cos(\omega_2 t + \varphi_2), \qquad \omega_2 = \omega_1 = \omega,\end{aligned} \tag{44}$$

where we have forced Eqs. (44) to appear to resemble Eqs. (43). For the two modes to have the same frequency is unusual; the two modes are then said to be "degenerate."

Example 7: Two-dimensional harmonic oscillator

In Fig. 1.7 we show a mass M that is free to move in the xy plane. It is coupled to the walls by two unstretched massless springs of spring constant K_1 oriented along x and by two unstretched massless springs of spring constant K_2 oriented along y. In the small-oscillations approximation, where we neglect x^2/a^2, y^2/a^2, and xy/a^2, we shall show that the x component of return force is due entirely to the two springs K_1. Similarly, the y component of return force is entirely due to the springs K_2. You can prove this by writing out the exact F_x and F_y and then discarding nonlinear terms. Here is an easier way to see it: Start at the equilibrium position of Fig. 1.7a. Mentally make a small displacement x of M in the $+x$ direction. The return force at this stage in the argument is given by inspection of Fig. 1.7:

$$F_x = -2K_1 x, \qquad F_y = 0.$$

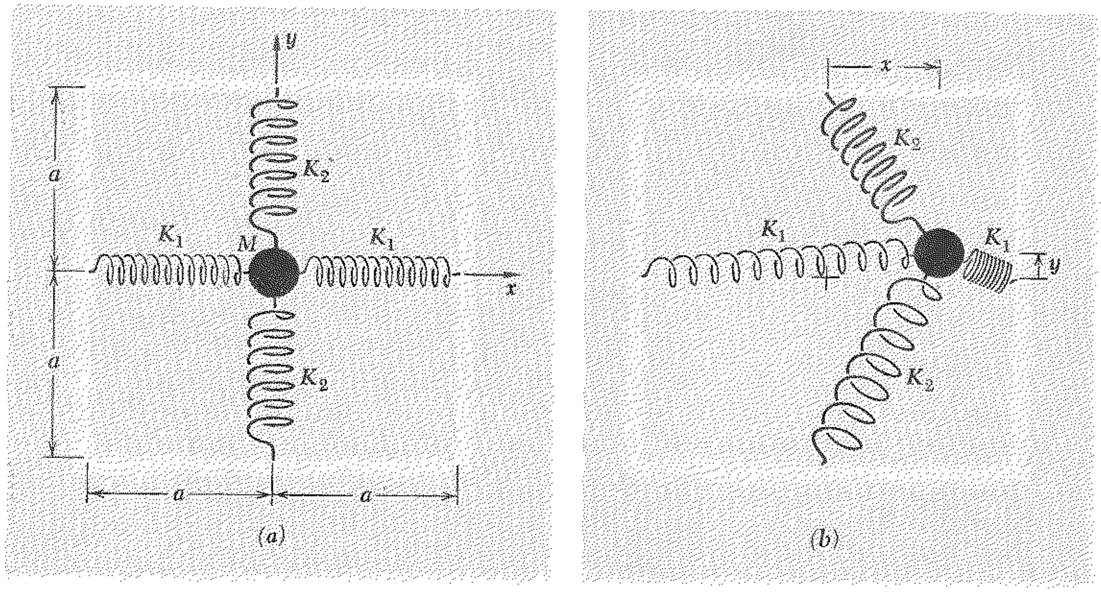


***Fig. 1.7** Two-dimensional harmonic oscillator. (a) Equilibrium. (b) General configuration.*

Next make a second small displacement y (starting at the terminus of the first displacement), this time in the $+y$ direction. The question of interest is whether F_x changes. The K_1 springs get longer by a small amount proportional to y^2. We neglect that. The K_2 springs change their length by an amount proportional to y (one gets shorter, the other longer), but the projection of their force on the x direction is proportional also to x. We neglect the product yx. Thus F_x is unchanged. A similar argument applies to F_y. Thus we obtain the two linear equations

$$M\frac{d^2x}{dt^2} = -2K_1x, \qquad \text{and} \qquad M\frac{d^2y}{dt^2} = -2K_2y, \tag{45}$$

which have the solutions

$$\begin{aligned} x &= A_1 \cos(\omega_1 t + \varphi_1), \qquad \omega_1^2 = \frac{2K_1}{M}, \\ y &= B_2 \cos(\omega_2 t + \varphi_2), \qquad \omega_2^2 = \frac{2K_2}{M}. \end{aligned} \tag{46}$$

We see that the x motion and y motion are uncoupled, and that each is a harmonic oscillation with its own frequency. Thus the x motion corresponds to one normal mode of oscillation, the y motion to the other. The x mode has amplitude A_1 and phase constant φ_1 that depend only on the initial values $x(0)$ and $\dot{x}(0)$, i.e., the x displacement and velocity at time $t = 0$. Similarly the y mode has amplitude B_2 and phase constant φ_2 that depend only on the initial values $y(0)$ and $\dot{y}(0)$.

Normal coordinates. Notice that our solution (46), which is completely general, is still not as general in appearance as Eqs. (43). That is because we were lucky! Our natural choice for x and y along the springs gave us the uncoupled equations (45), each of which corresponds to one of the modes. In terms of Eq. (43), we came out with ψ_a luckily chosen so that A_2 came out identically zero and with ψ_b chosen so that B_1 came out identically zero. Our fortunate choice of coordinates gave us what are called *normal coordinates;* in this example the normal coordinates are x and y.

Suppose we had not been so lucky or so wise. Suppose we had used a coordinate system x' and y' related to x and y by a rotation through angle α, as shown in Fig. 1.8. By inspection of the figure we see that the normal coordinate x is a linear combination of the coordinates x' and y', as is the other normal coordinate, y. If we had used the "dumb" coordinates x' and y' instead of the "smart" coordinates x and y, we would have obtained two "coupled" differential equations, with both x' and y' appearing in each equation, rather than the uncoupled equations (5).

In most problems involving two degrees of freedom it is not easy to find the normal coordinates "by inspection," as we did in the present example. Thus the equations of motion of the different degrees of freedom are usually coupled equations. One method of solving these two coupled differential equations is to search for new variables that are linear combinations of the original "dumb" coordinates such that the new variables satisfy uncoupled equations of motion. The new variables are then called "normal coordinates." In the present example we know how to find the

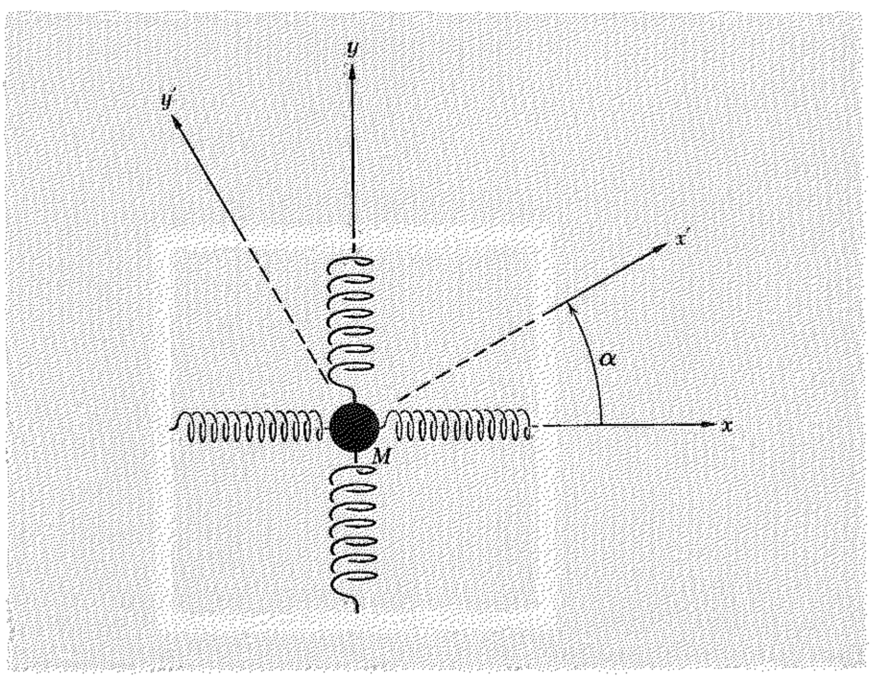


Fig. 1.8 Rotation of coordinates.

normal coordinates, given the "dumb" coordinates x' and y'. Simply rotate the coordinate system so as to obtain x and y, each of which is a linear combination of x' and y'. In a more general problem, we would have to use a more general linear transformation of coordinates than can be obtained by a simple rotation. That would be the case if, for example, the pairs of springs in Fig. 1.7 were not orthogonal.

Systematic solution for modes. Without considering any specific physical system, we assume that we have found two coupled first-order linear homogeneous equations in the "dumb" coordinates x and y:

$$\frac{d^2x}{dt^2} = -a_{11}x - a_{12}y \tag{47}$$

$$\frac{d^2y}{dt^2} = -a_{21}x - a_{22}y. \tag{48}$$

Now we simply *assume* that we have oscillation in a single normal mode. That means we assume that both degrees of freedom, namely x and y, oscillate with harmonic motion with the *same frequency and same phase constant.* Thus we *assume* we have

$$x = A\cos(\omega t + \varphi), \qquad y = B\cos(\omega t + \varphi), \tag{49}$$

with ω unknown and B/A unknown at this stage. Then we have

$$\frac{d^2x}{dt^2} = -\omega^2 x, \qquad \frac{d^2y}{dt^2} = -\omega^2 y. \tag{50}$$

Substituting Eq. (50) into Eqs. (47) and (48) and rearranging, we obtain two homogeneous linear equations in x and y:

$$(a_{11} - \omega^2)x + a_{12}y = 0, \tag{51}$$

$$a_{21}x + (a_{22} - \omega^2)y = 0. \tag{52}$$

Equations (51) and (52) each give the ratio y/x:

$$\frac{y}{x} = \frac{\omega^2 - a_{11}}{a_{12}}, \tag{53}$$

$$\frac{y}{x} = \frac{a_{21}}{\omega^2 - a_{22}}. \tag{54}$$

For consistency, we need to have Eqs. (53) and (54) give the same result. Thus we need the condition

$$\frac{\omega^2 - a_{11}}{a_{12}} = \frac{a_{21}}{\omega^2 - a_{22}},$$

i.e.,

$$(a_{11} - \omega^2)(a_{22} - \omega^2) - a_{21}a_{12} = 0. \tag{55}$$

Another way to write Eq. (55) is to say that the determinant of coefficients of the linear homogeneous equations (51) and (52) must vanish:

$$\begin{vmatrix} a_{11}-\omega^2 & a_{12} \\ a_{21} & a_{22}-\omega^2 \end{vmatrix} \equiv (a_{11}-\omega^2)(a_{22}-\omega^2) - a_{21}a_{12} = 0. \quad (56)$$

Equation (55) or (56) is a quadratic equation in the variable ω^2. It has two solutions, which we call ω_1^2 and ω_2^2. Thus we have found that if we assume we have oscillation in a single mode, there are exactly two ways that that assumption can be realized. Frequency ω_1 is the frequency of mode 1; ω_2 is that of mode 2. The shape or configuration of x and y in mode 1 is obtained by substituting $\omega^2 = \omega_1^2$ back into either one of Eqs. (53) and (54). [They are equivalent, because of Eq. (56).] Thus

$$\left(\frac{y}{x}\right)_{\text{mode 1}} = \left(\frac{B}{A}\right)_{\text{mode 1}} = \frac{B_1}{A_1} = \frac{\omega_1^2 - a_{11}}{a_{12}}. \quad (57a)$$

Similarly,

$$\left(\frac{y}{x}\right)_{\text{mode 2}} = \left(\frac{B}{A}\right)_{\text{mode 2}} = \frac{B_2}{A_2} = \frac{\omega_2^2 - a_{11}}{a_{12}}. \quad (57b)$$

Once we have found the mode frequencies ω_1 and ω_2 and the amplitude ratios B_1/A_1 and B_2/A_2, we can write down the most general superposition of the two modes as follows:

$$x(t) = x_1(t) + x_2(t) = A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \varphi_2), \quad (58)$$

$$\begin{aligned} y(t) &= \frac{B_1}{A_1} A_1 \cos(\omega_1 t + \varphi_1) + \frac{B_2}{A_2} A_2 \cos(\omega_2 t + \varphi_2) \\ &= B_1 \cos(\omega_1 t + \varphi_1) + B_2 \cos(\omega_2 t + \varphi_2). \end{aligned} \quad (59)$$

Notice that, whereas we chose A_1, φ_1, A_2, and φ_2 with complete freedom in Eq. (58), we had no freedom at all left when we came to write the constants in Eq. (59), because φ_1 and φ_2 were already fixed and because we had to satisfy Eqs. (57).

The most general solution of Eqs. (47) and (48) consists of a superposition of any two independent solutions which satisfies the four initial conditions given by $x(0)$, $\dot{x}(0)$, $y(0)$, and $\dot{y}(0)$. A superposition of the two normal modes, with the four constants A_1, φ_1, A_2, and φ_2 determined by the four initial conditions, is such a solution. Thus the general solution can be (although it need not be) written as a superposition of the modes.

Example 8: Longitudinal oscillations of two coupled masses

The system is shown in Fig. 1.9. The two masses M slide on a frictionless table. The three springs are massless and identical, each with spring constant K. We will let the reader do the systematic solution (Prob. 1.23), but here let us try to *guess* the normal modes. We know there must be two

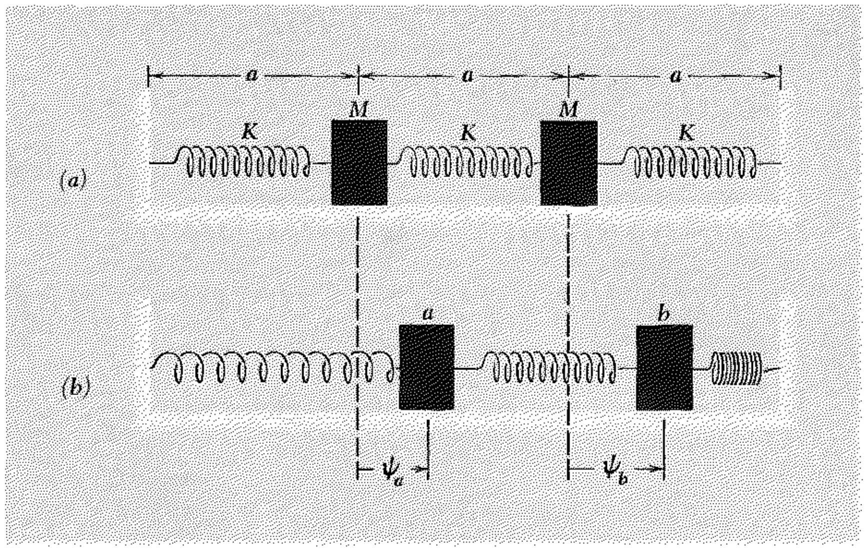


Fig. 1.9 Longitudinal oscillations. (a) Equilibrium. (b) General configuration.

modes, since there are two degrees of freedom. In a mode, each moving part (each mass) oscillates with harmonic motion. This means that each moving part oscillates with the same frequency, and thus *the return force per unit displacement per unit mass is the same for both masses.* (We learned in Sec. 1.2 that ω^2 is the return force per unit displacement per unit mass. That holds for each moving part, whether it is a single isolated system with one degree of freedom or is part of a larger system. The only requirement is that the motion be harmonic motion with a single frequency.)

In the present example the masses are equal. We need therefore only search for configurations that have the same return force per unit displacement for both masses. Let us guess that the displacements may be the same, and see if that works: Suppose we start at the equilibrium position and then displace both masses by the same amount to the right. Is the return force the same on each mass? Notice that the central spring has the same length as it had at equilibrium, so that it exerts no force on either mass. The left-hand mass is pulled to the left because the left-hand spring is extended. The right-hand mass is pushed to the left with the *same* force, because the right-hand spring is compressed by the same amount. We have therefore discovered one mode!

$$\text{Mode 1:} \qquad \psi_a(t) = \psi_b(t), \qquad \omega_1^2 = \frac{K}{M}. \tag{60}$$

The frequency $\omega_1^2 = K/M$ in Eq. (60) follows from the fact that each mass oscillates just as it would if the central spring were removed.

Now let us try to guess the second mode. From the symmetry, we guess that if a and b move oppositely we may have a mode. If a moves a distance ψ_a to the right and b moves an equal distance to the left, each has the same return force. Thus the second mode has $\psi_b = -\psi_a$. The frequency ω_2 can be found by considering a single mass and finding its return force per unit displacement per unit mass. Consider the left-hand mass a. It is pulled to the left by the left-hand spring with a force $F_z = -K\psi_a$. It is pushed to the left by the middle spring with a force $F_z = -2K\psi_a$. (The factor of two occurs because the central spring is compressed by an amount $2\psi_a$.) Thus the net force for a displacement ψ_a is $-3K\psi_a$, and the return force per unit displacement per unit mass is $3K/M$:

$$\text{Mode 2:} \qquad \psi_a = -\psi_b, \qquad \omega_2{}^2 = \frac{3K}{M}. \tag{61}$$

The modes are shown in Fig. 1.10.

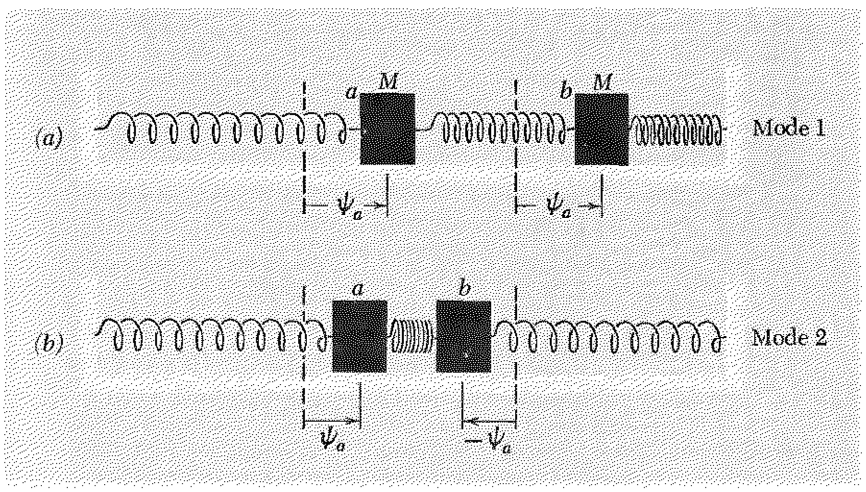


Fig. 1.10 Normal modes of longitudinal oscillation. (a) Mode with lower frequency. (b) Mode with higher frequency.

We shall solve this problem once more, using the method of searching for normal coordinates, i.e., "smart" coordinates. The "smart" coordinates are always a linear combination of ordinary "dumb" coordinates, such that instead of two coupled linear equations, one obtains two uncoupled equations. From Fig. 1.9b, we easily see that the equations of motion for a general configuration are

$$M\frac{d^2\psi_a}{dt^2} = -K\psi_a + K(\psi_b - \psi_a), \tag{62}$$

$$M\frac{d^2\psi_b}{dt^2} = -K(\psi_b - \psi_a) - K\psi_b. \tag{63}$$

By inspection of these equations of motion, we see that alternately adding and subtracting these equations will produce the desired uncoupled equations. Adding Eqs. (62) and (63), we obtain

$$M\frac{d^2}{dt^2}(\psi_a + \psi_b) = -K(\psi_a + \psi_b). \tag{64}$$

Subtracting Eq. (63) from Eq. (62), we obtain

$$M\frac{d^2(\psi_a - \psi_b)}{dt^2} = -3K(\psi_a - \psi_b). \tag{65}$$

Equations (64) and (65) are uncoupled equations in the variables $\psi_a + \psi_b$ and $\psi_a - \psi_b$. They have the solutions

$$\psi_a + \psi_b \equiv \psi_1(t) = A_1 \cos(\omega_1 t + \varphi_1), \qquad \omega_1^2 = \frac{K}{M}, \tag{66}$$

$$\psi_a - \psi_b \equiv \psi_2(t) = A_2 \cos(\omega_2 t + \varphi_2), \qquad \omega_2^2 = \frac{3K}{M}, \tag{67}$$

where A_1 and φ_1 are the amplitude and phase constant of mode 1 and where A_2 and φ_2 are the amplitude and phase constant of mode 2. We see that $\psi_1(t)$ corresponds to the motion of the center of mass, since $\frac{1}{2}(\psi_a + \psi_b)$ is the position of the center of mass. (We could have divided Eq. (64) by 2 and defined ψ_1 to be the position of the center of mass. The proportionality factor of $\frac{1}{2}$ is not of much interest.) We see that ψ_2 is the compression of the central spring, or (what amounts to the same thing) it is the relative displacement of the two masses. If we had been smart enough, we might have chosen ψ_1 and ψ_2 to start with, since the motion of the center of mass and the "internal motion" (relative motion of the two particles) are physically interesting variables. In many cases it is not so easy to find a simple physical meaning for the normal coordinates. Thus we shall usually stick with our original "dumb" coordinates even after finding the modes, simply because we understand them best.

In the present problem we have found the normal coordinates ψ_1 and ψ_2. Let us go back to our more familiar coordinates ψ_a and ψ_b. Solving Eqs. (66) and (67), we find

$$2\psi_a = A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \varphi_2) \tag{68}$$

$$2\psi_b = A_1 \cos(\omega_1 t + \varphi_1) - A_2 \cos(\omega_2 t + \varphi_2). \tag{69}$$

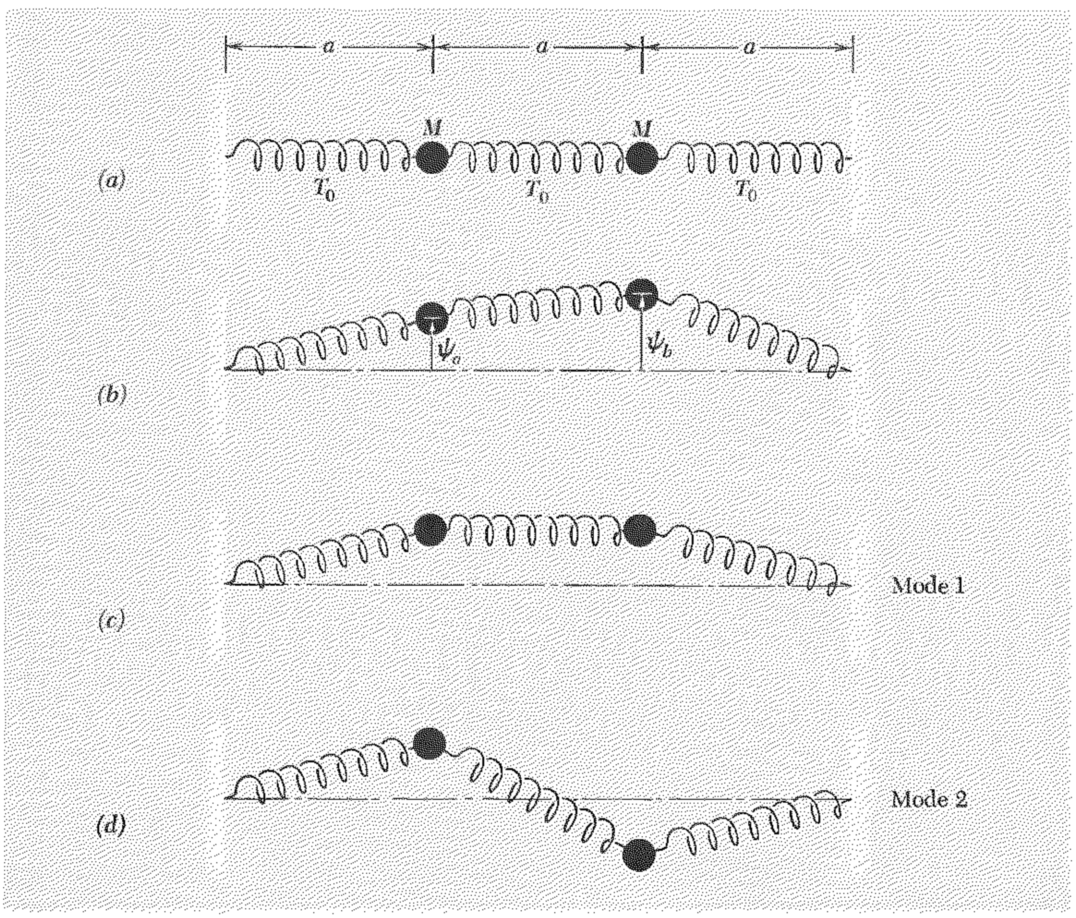
Notice that if we have a motion that is purely mode 1, then A_2 is zero, and, according to Eqs. (68) and (69), we have $\psi_b = \psi_a$. Similarly, in mode 2 we have $A_1 = 0$ and $\psi_b = -\psi_a$. That is what we found before [in Eqs. (60) and (61)].

Example 9: Transverse oscillations of two coupled masses

The system is shown in Fig. 1.11. The oscillations are assumed to be confined to the plane of the paper. Therefore there are just two degrees of freedom. The three identical massless springs have a relaxed length a_0 that is less than the equilibrium spacing a of the masses. Thus they are all stretched. When the system is at its equilibrium configuration (Fig. 1.11*a*), the springs have tension T_0.

Because of the symmetry of the system, the modes are easy to guess. They are shown in Fig. 1.11. The lower mode (the one with the lower frequency, i.e., the one with the smaller return force per unit displacement per unit mass for each of the masses) has a shape (Fig. 1.11*c*) such that the center spring is never compressed or extended. The frequency is thus obtained by considering either one of the masses separately, with the return force provided only by the spring that connects it to the wall. For either the slinky approximation (unstretched spring length of zero) or the small-oscillations approximation (displacements very small compared with the spacing a), we shall show presently that a displacement ψ_a of the left-hand

Fig. 1.11 Transverse oscillations. (a) Equilibrium. (b) General configuration. (c) Mode with lower frequency. (d) Mode with higher frequency.



mass causes the left-hand spring to exert a return force of $T_0(\psi_a/a)$. Hence, in this mode the return force per unit displacement per unit mass, $\omega_1{}^2$, is given by

$$\text{Mode 1:} \qquad \omega_1{}^2 = \frac{T_0}{Ma}, \qquad \frac{\psi_b}{\psi_a} = +1. \tag{70}$$

We see this as follows. First consider the slinky approximation (Sec. 1.2). In this approximation, the tension T is larger than T_0 by the factor l/a, where l is the spring length and a is the length at equilibrium (Fig. 1.11*a*). The spring exerts a transverse return force equal to the tension T times the sine of the angle between the spring and the equilibrium axis of the springs, i.e., the return force is $T(\psi_a/l)$. But $T = T_0(l/a)$. Thus the return force is $T_0(\psi_a/a)$, and this gives Eq. (70). Next consider the small-oscillations approximation (Sec. 1.2). In that approximation, the increase in length of the spring is neglected, because it differs from the equilibrium length a only by a quantity of order $a(\psi_a/a)^2$, and therefore the increase in tension also is neglected. The tension is thus T_0 when the displacement is ψ_a. The return force is equal to the tension T_0 times the sine of the angle between the spring and the equilibrium axis. This angle may be taken to be a "small" angle, since the oscillations are small. Then the angle (in radians) and its sine are equal, and both are equal to ψ_a/a. Thus the return force is $T_0(\psi_a/a)$. This gives Eq. (70).

Similarly, we can obtain the frequency for mode 2 (Fig. 1.11*d*) as follows: Consider the left-hand mass. The left-hand spring contributes a return force per unit displacement per unit mass of T_0/Ma, as we have just seen in considering mode 1. In mode 2 the center spring is "helping" the left-hand spring, and in fact it is providing twice as great a return force as is the left-hand spring. This is easily seen in the small-oscillations approximation: The spring tension is T_0 for both springs, but the center spring makes twice as large an angle with the axis as does the end spring, so that it gives twice as large a transverse force component. The total return force per unit displacement per unit mass, $\omega_2{}^2$, is thus given by

$$\text{Mode 2:} \qquad \omega_2{}^2 = \frac{T_0}{Ma} + \frac{2T_0}{Ma} = \frac{3T_0}{Ma}, \qquad \frac{\psi_b}{\psi_a} = -1. \tag{71}$$

Notice that in the slinky approximation, where the relation $T_0 = K(a - a_0)$ becomes $T_0 = Ka$, the frequencies of the modes of transverse oscillation [Eqs. (70) and (71)] are the same as those for longitudinal oscillation [Eqs. (60) and (61)]. Thus we have a form of degeneracy. This degeneracy does not occur for the small-oscillation approximation, where a_0 is not negligible compared with a.

If the modes had not been so easy to guess, we would have written down the equations of motion of the two masses a and b and then proceeded with the equations, rather than with a mental picture of the physical system itself. We shall let you do that (Prob. 1.20).

Example 10: Two coupled LC circuits

Consider the system shown in Fig. 1.12. Let us find the equations of "motion"—motion of the charges in this case. The electromotive force (emf) across the left-hand inductance is $L\,dI_a/dt$. A positive charge Q_1 on the left-hand capacitor gives an emf $C^{-1}Q_1$ that tends to increase I_a (with our sign conventions). A positive charge Q_2 on the middle capacitor gives an emf $C^{-1}Q_2$ that tends to decrease I_a. Thus we have for the complete contribution to $L\,dI_a/dt$

$$\frac{L\,dI_a}{dt} = C^{-1}Q_1 - C^{-1}Q_2. \tag{72}$$

Similarly,

$$\frac{L\,dI_b}{dt} = C^{-1}Q_2 - C^{-1}Q_3. \tag{73}$$

As in Sec. 1.2, we will express the configuration of the system in terms of currents rather than charges. To do this, we differentiate Eqs. (72) and (73) with respect to time and use conservation of charge. Differentiating gives

$$L\frac{d^2I_a}{dt^2} = C^{-1}\frac{dQ_1}{dt} - C^{-1}\frac{dQ_2}{dt}, \tag{74}$$

$$L\frac{d^2I_b}{dt^2} = C^{-1}\frac{dQ_2}{dt} - C^{-1}\frac{dQ_3}{dt}. \tag{75}$$

Charge conservation gives

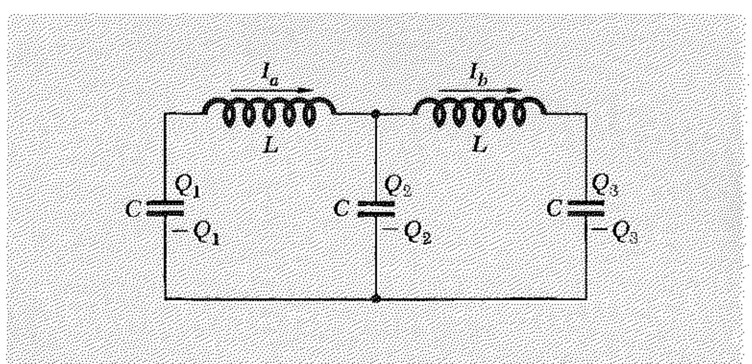
$$\frac{dQ_1}{dt} = -I_a, \qquad \frac{dQ_2}{dt} = I_a - I_b, \qquad \frac{dQ_3}{dt} = I_b. \tag{76}$$

Substituting Eqs. (76) into Eqs. (74) and (75), we obtain the coupled equations of motion

$$L\frac{d^2I_a}{dt^2} = -C^{-1}I_a + C^{-1}(I_b - I_a) \tag{77}$$

$$L\frac{d^2I_b}{dt^2} = -C^{-1}(I_b - I_a) - C^{-1}I_b. \tag{78}$$

Fig. 1.12 Two coupled LC circuits. General configuration of charges and currents. The arrows give sign conventions for positive currents.



Now that we have the two equations of motion we want to find the two normal modes. These can be found by searching for normal coordinates, by guessing, or by the systematic method (see Prob. 1.21). One finds

$$\begin{aligned} &\text{Mode 1:} \qquad I_a = I_b, \qquad \omega_1{}^2 = \frac{C^{-1}}{L}. \\ &\text{Mode 2:} \qquad I_a = -I_b, \qquad \omega_2{}^2 = \frac{3C^{-1}}{L}. \end{aligned} \tag{79}$$

Notice that in mode 1 the center capacitor never acquires any charge, and it could be removed without affecting the motion of the charges. Also, in mode 1 the charges Q_1 and Q_3 are always equal in magnitude and opposite in sign. In mode 2 the charges Q_1 and Q_3 are always equal in both magnitude and sign, and Q_2 has twice that magnitude, but opposite sign.

We purposely chose the three examples (8–10) of longitudinal oscillations (Fig. 1.9), transverse oscillations (Fig. 1.11), and coupled LC circuits (Fig. 1.12) to have the same spatial symmetry and to give equations of motion and normal modes with the same mathematical form. We also chose these examples to be the natural extensions (to two degrees of freedom) of the similar systems with one degree of freedom that we considered in Examples 2–4 in Sec. 1.2, as shown in Figs. 1.3, 1.4, and 1.5. In Chap. 2 we shall extend these same three examples to an arbitrarily large number of degrees of freedom.

1.5 Beats

There are many physical phenomena where the motion of a given moving part is a superposition of two harmonic oscillations having different angular frequencies ω_1 and ω_2. For example, the two harmonic oscillations may correspond to the two normal modes of a system having two degrees of freedom. As a contrasting example, the two harmonic oscillations may be due to driving forces produced by two independently oscillating uncoupled systems. This sort of situation is illustrated by two tuning forks of different frequencies. Each produces its own "note" by causing harmonically oscillating pressure variations at the fork, which radiate through the air as sound waves. The motion induced in your eardrum is a superposition of two harmonic oscillations.

In all these examples, the mathematics is the same. For simplicity we assume that the two harmonic oscillations have the same amplitude. We also assume that the two oscillations have the same phase constant, which we take to be zero. Then we write the superposition ψ of the two harmonic oscillations ψ_1 and ψ_2:

$$\psi_1 = A \cos \omega_1 t, \qquad \psi_2 = A \cos \omega_2 t, \tag{80}$$

$$\psi = \psi_1 + \psi_2 = A \cos \omega_1 t + A \cos \omega_2 t. \tag{81}$$

Modulation. We shall now recast Eq. (81) into an interesting form. We define an "average" angular frequency ω_{av} and a "modulation" angular frequency ω_{mod}:

$$\omega_{av} \equiv \tfrac{1}{2}(\omega_1 + \omega_2), \qquad \omega_{mod} \equiv \tfrac{1}{2}(\omega_1 - \omega_2). \tag{82}$$

The sum and difference of these give

$$\omega_1 = \omega_{av} + \omega_{mod}, \qquad \omega_2 = \omega_{av} - \omega_{mod}. \tag{83}$$

Then we may write Eq. (81) in terms of ω_{av} and ω_{mod}:

$$\begin{aligned} \psi &= A \cos \omega_1 t + A \cos \omega_2 t \\ &= A \cos (\omega_{av} t + \omega_{mod} t) + A \cos (\omega_{av} t - \omega_{mod} t) \\ &= [2A \cos \omega_{mod} t] \cos \omega_{av} t, \end{aligned}$$

i.e.,

$$\psi = A_{mod}(t) \cos \omega_{av} t, \tag{84}$$

where

$$A_{mod}(t) = 2A \cos \omega_{mod} t. \tag{85}$$

We can think of Eqs. (84) and (85) as representing an oscillation at angular frequency ω_{av}, with an amplitude A_{mod} that is not constant but rather varies with time according to Eq. (85). Equations (84) and (85) are exact. However, it is most useful to write the superposition, Eq. (81), in the form of Eqs. (84) and (85) when ω_1 and ω_2 are of comparable magnitude. Then the modulation frequency is small in magnitude compared with the average frequency:

$$\omega_1 \approx \omega_2; \qquad \omega_{mod} \ll \omega_{av}.$$

In that case, the modulation amplitude, $A_{mod}(t)$, varies only slightly during several of the so-called "fast" oscillations of $\cos \omega_{av} t$, and therefore Eq. (84) corresponds to "almost harmonic" oscillation at frequency ω_{av}. Of course, if A_{mod} is exactly constant, Eq. (84) represents exact harmonic oscillation at angular frequency ω_{av}. Then $\omega_{av} = \omega_1 = \omega_2$, since A_{mod} is only constant if ω_{mod} is zero. If ω_1 and ω_2 differ only slightly, the superposition of the two (exactly harmonic) oscillations ω_1 and ω_2 is called an "almost harmonic" or "almost monochromatic" oscillation of frequency ω_{av} with a slowly varying amplitude.

Almost harmonic oscillation. This is our first example of a very important and very general result that we will encounter many times: A linear superposition of two or more exactly harmonic oscillations having different frequencies (and different amplitudes and phase constants), with all the frequencies lying in a relatively narrow range or "band" of frequencies, gives a resultant oscillation that is "almost" a harmonic oscillation, with a frequency ω_{av} that lies somewhere in the band of the "component" oscillations that make up the superposition. The resultant motion is not exactly a harmonic oscillation because the amplitude and phase constant are not exactly constant, but only "almost constant." Their variation is negligible during one cycle of oscillation at the average "fast" frequency ω_{av}, provided that the frequency range or "bandwidth" of the component harmonic oscillations is small compared with ω_{av}. (We shall prove these remarks in Chap. 6.)

Some physical examples of beats follow:

Example 11: Beats produced by two tuning forks

When a sound wave reaches your ear, it produces a variation in air pressure at the eardrum. Let ψ_1 and ψ_2 represent the respective contributions to the gauge pressure produced outside your eardrum by two tuning forks, numbered 1 and 2. (The gauge pressure is just the pressure on the outer surface of your eardrum minus the pressure on the inner surface; the pressure on the inner surface is normal atmospheric pressure. This pressure difference provides the driving force to drive the eardrum.)

If both forks are struck equally hard at the same time and are held at the same distance from the eardrum, the amplitudes and phase constants for the gauge pressures ψ_1 and ψ_2 are the same, and thus Eq. (80) correctly represents the two pressure contributions. The total pressure (which gives the total force on the drum) is the superposition $\psi = \psi_1 + \psi_2$ of the contributions from the two forks. It is given either by Eq. (81) or by Eqs. (84) and (85). If the frequencies of the two forks, ν_1 and ν_2, differ by more than about 6% of their average value, then your ear and brain ordinarily prefer Eq. (81). That is, you "hear" the total sound as two separate notes with slightly different pitches. For example, if ν_2 is $\frac{5}{4}$ times ν_1, you hear two notes with an interval of a "major third." If ν_2 is $1.06\nu_1$, you hear ν_2 as a note "one half-tone higher" in pitch than ν_1. However, if ν_1 and ν_2 differ by less than about 10 cps, your ear (plus brain) no longer easily recognizes them as different notes. (A musician's trained ear may do much better.) Then a superposition of the two is not heard as a "chord" made up of the two notes ν_1 and ν_2, but rather as a single pitch of frequency ν_{av} with a slowly varying amplitude A_{mod}, just as given by Eqs. (84) and (85).

Square-law detector. The modulation amplitude A_{mod} oscillates at the modulation angular frequency ω_{mod}. Whenever $\omega_{mod}t$ has increased by an amount 2π (radians of phase), the amplitude A_{mod} has gone through one complete cycle of oscillation (i.e., the "slow" oscillation at the modulation frequency) and has returned to its original value. At two times during one cycle, A_{mod} is zero. At those times, the ear doesn't hear anything—there is no sound. In between the silences, you hear a sound at the average pitch. Since $\cos \omega_{mod}t$ goes from zero to $+1$, to zero, to -1, to zero, to

+1, etc., we see that A_{mod} has opposite signs at successive loud times. Nevertheless, your ear does not recognize "two kinds" of loud times, as you will discover if you perform the experiment with tuning forks. Thus your ear (plus brain) does not distinguish positive from negative values of A_{mod}. It only distinguishes whether the magnitude of A_{mod} is large ("loud") or small ("soft"), that is, whether the *square* of A_{mod} is large or small. For that reason, your ear (plus brain) is sometimes said to be a *square-law detector.* Since $A_{\text{mod}}{}^2$ has *two* maxima for every modulation cycle (during which $\omega_{\text{mod}}t$ increases by 2π), the repetition rate for the sequence "loud, soft, loud, soft, loud, soft, . . ." is twice the modulation frequency. This repetition rate of large values of $A_{\text{mod}}{}^2$ is called the *beat frequency:*

$$\omega_{\text{beat}} = 2\omega_{\text{mod}} = \omega_1 - \omega_2. \tag{86}$$

We can see this algebraically as follows:

$$A_{\text{mod}}(t) = 2A \cos \omega_{\text{mod}}t.$$
$$[A_{\text{mod}}(t)]^2 = 4A^2 \cos^2 \omega_{\text{mod}}t;$$

but

$$\cos^2 \theta = \tfrac{1}{2}[\cos^2 \theta + \sin^2 \theta + \cos^2 \theta - \sin^2 \theta] = \tfrac{1}{2}[1 + \cos 2\theta].$$

Thus

$$[A_{\text{mod}}(t)]^2 = 2A^2[1 + \cos 2\omega_{\text{mod}}t],$$

i.e.,

$$(A_{\text{mod}})^2 = 2A^2[1 + \cos \omega_{\text{beat}}t]. \tag{87}$$

Thus $A_{\text{mod}}{}^2$ oscillates about its average value at twice the modulation frequency, i.e., at the beat frequency, $\omega_1 - \omega_2$.

The superposition of two harmonic oscillations with nearly equal frequencies to produce beats is illustrated in Fig. 1.13.

Example 12: Beats between two sources of visible light

In 1955, Forrester, Gudmundsen, and Johnson performed a beautiful experiment showing beats between two independent sources of visible light with nearly the same frequency.† The light sources were gas discharge tubes containing freely decaying mercury atoms with an average frequency of $\nu_{\text{av}} = 5.49 \times 10^{14}$ cps, corresponding to the bright "green line" of mercury. The atoms were placed in a magnetic field. This caused the green radiation to "split" into two neighboring frequencies, with the frequency difference proportional to the magnetic field. The beat frequency was $\nu_1 - \nu_2 \approx 10^{10}$ cps. This is a typical "radar" or "microwave" frequency.

† A. T. Forrester, R. A. Gudmundsen, and P. O. Johnson, "Photoelectric mixing of incoherent light," *Phys. Rev.* 99, 1691 (1955).

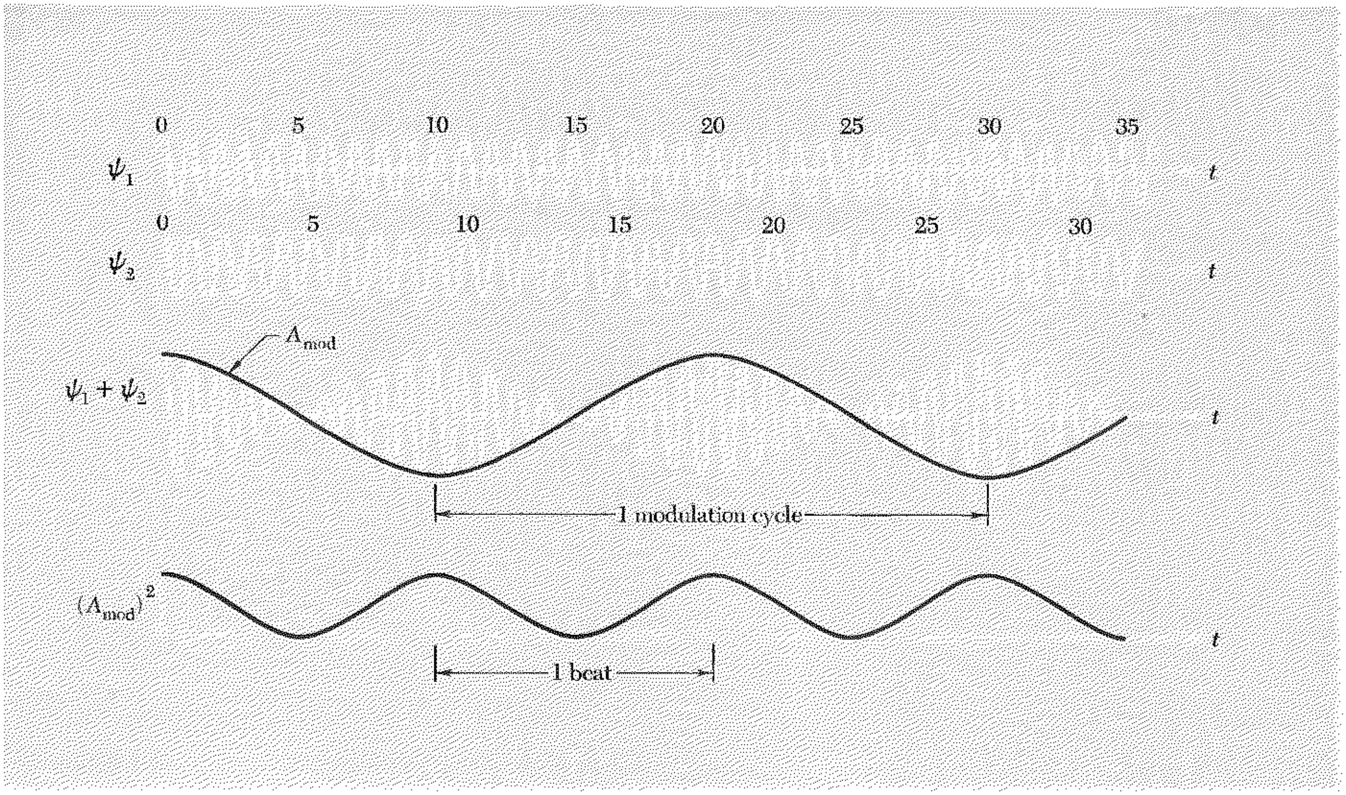


***Fig. 1.13** Beats. ψ_1 and ψ_2 are the pressure variations at your ear produced by two tuning forks with frequency ratio $\nu_1/\nu_2 = 10/9$. The total pressure is the superposition $\psi_1 + \psi_2$, which is an "almost harmonic" oscillation at frequency ν_{av} with slowly varying amplitude $A_{mod}(t)$. The loudness is proportional to $(A_{mod})^2$ and consists of a constant (average value) plus a sinusoidal variation at the beat frequency. The beat frequency is twice the modulation frequency.*

Their detector used the photoelectric effect to give an output electric current proportional to the square of the modulation amplitude of the resultant electric field in the light wave. Thus the detector was a square-law detector. The output of their detector showed a time variation similar to the "loudness," $A_{mod}{}^2$, in Fig. 1.13.

Example 13: Beats between the two normal modes of two weakly coupled identical oscillators

Consider the system of two identical pendulums coupled by a spring shown in Fig. 1.14. The normal modes are easily guessed by analogy with the longitudinal oscillations of the identical masses studied in Sec. 1.4. In mode 1 we have $\psi_a = \psi_b$. The coupling spring could just as well be removed; the return force is entirely due to gravity. The return force per unit displacement per unit mass (assuming small-oscillation amplitudes, for which we have a linear restoring force) is $Mg\theta/(l\theta)M = g/l$:

$$\text{Mode 1:} \qquad \omega_1{}^2 = \frac{g}{l}, \qquad \psi_a = \psi_b. \tag{88}$$

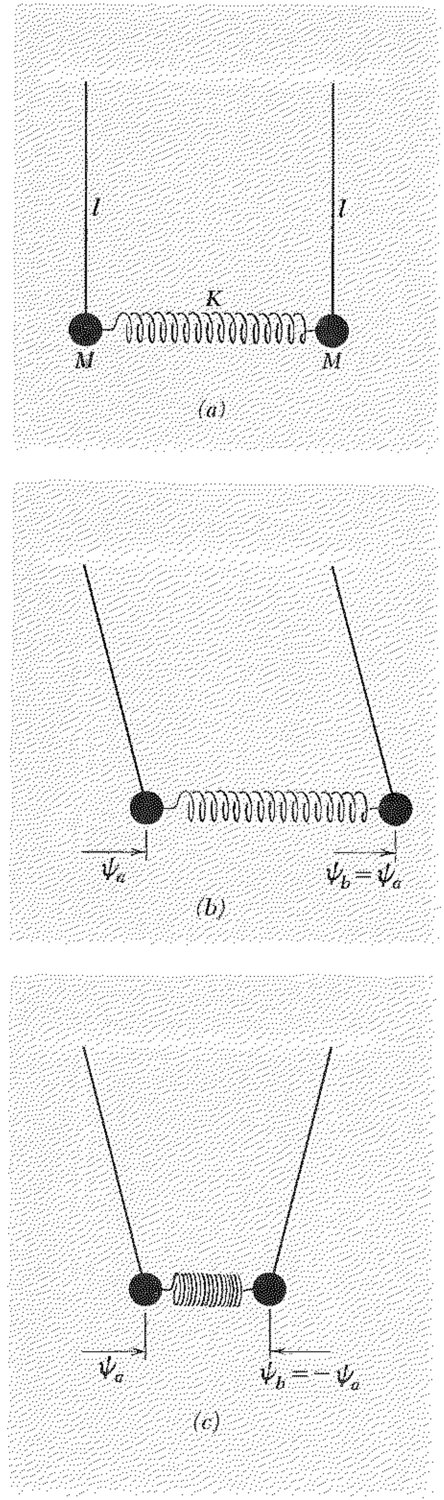


Fig. 1.14 ***Coupled identical pendulums. (a) Equilibrium configuration. (b) Mode with lower frequency. (c) Mode with higher frequency.***

In mode 2 we have $\psi_a = -\psi_b$. Consider the left-hand bob. The return force due to the spring is $2K\psi_a$. (The factor of 2 results from the fact that the spring is compressed by an amount $2\psi_a$ in this mode when bob a is displaced by an amount ψ_a.) The return force due to gravity is $Mg\theta = Mg\psi_a/l$. The spring and gravity both act with the same sign. Thus the total return force per unit displacement per unit mass is

$$\text{Mode 2:} \qquad \omega_2^2 = \frac{g}{l} + \frac{2K}{M}, \qquad \psi_a = -\psi_b. \tag{89}$$

We now wish to study "beats between the two modes" of this system. What does that mean? Each mode is a harmonic oscillation with a given frequency. The general motion of pendulum a is given by a superposition of the two modes:

$$\psi_a(t) = \psi_1(t) + \psi_2(t).$$

Thus, $\psi_a(t)$ will look like the superposition $\psi_1 + \psi_2$ in Fig. 1.13 if the mode frequencies are nearly the same (and if the amplitudes of the two modes are the same). Then we say that the motion of pendulum a exhibits beats. (Of course pendulum b will also exhibit beats, as we shall see.) Any system of two degrees of freedom can exhibit beats, but the system we have chosen is convenient because we can easily make the beat frequency $\nu_1 - \nu_2$ small compared with the average frequency by using a sufficiently weak spring or by making the mass M large. [To see this, compare Eqs. (88) and (89).]

What do the beats look like? According to our discussion in Sec. 1.4, the displacements of the bobs, ψ_a and ψ_b, can be expressed in terms of the normal coordinates ψ_1 and ψ_2 by the general superposition

$$\begin{aligned} \psi_a &= \psi_1 + \psi_2 = A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \varphi_2), \\ \psi_b &= \psi_1 - \psi_2 = A_1 \cos(\omega_1 t + \varphi_1) - A_2 \cos(\omega_2 t + \varphi_2). \end{aligned} \tag{90}$$

By analogy with the tuning forks, we will get the largest beat effect if the two modes are present with equal amplitudes. (If either A_1 or A_2 is nearly zero compared to the other, there is virtually no beat effect, since (approximately) only one harmonic oscillation is present. Both oscillations should have approximately equal amplitudes to produce strong beats.) Therefore we take $A_1 = A_2 = A$. The choice of phase constants φ_1 and φ_2 corresponds to the initial conditions, as we shall see. By analogy with our example of the tuning forks, we take $\varphi_1 = \varphi_2 = 0$. With these choices for A_1, A_2, φ_1, and φ_2, Eqs. (90) give

$$\psi_a(t) = A\cos\omega_1 t + A\cos\omega_2 t, \qquad \psi_b(t) = A\cos\omega_1 t - A\cos\omega_2 t. \tag{91}$$

The velocities of the bobs are given by

$$\dot{\psi}_a(t) \equiv \frac{d\psi_a}{dt} = -\omega_1 A \sin \omega_1 t - \omega_2 A \sin \omega_2 t,$$
$$\dot{\psi}_b(t) \equiv \frac{d\psi_b}{dt} = -\omega_1 A \sin \omega_1 t + \omega_2 A \sin \omega_2 t. \tag{92}$$

In order to see how to excite the two modes in just such a way as to get oscillations corresponding to Eq. (91), let us consider the *initial conditions* at time $t = 0$. According to Eqs. (91) and (92), the initial displacements and velocities of the bobs are given by

$$\psi_a(0) = 2A, \qquad \psi_b(0) = 0; \qquad \dot{\psi}_a(0) = 0, \qquad \dot{\psi}_b(0) = 0.$$

Therefore we hold bob a at displacement $2A$, bob b at zero, and release both bobs from rest at the same time, which we call $t = 0$.

After that we just watch. (You should do this experiment yourself. You need two cans of soup, a slinky, and some string. See Home Experiment 1.8.) A fascinating process unfolds. Gradually the oscillation amplitude of pendulum a decreases and that of pendulum b increases, until eventually pendulum a is resting and pendulum b is oscillating with the amplitude and energy that pendulum a started out with. (We are neglecting frictional forces.) The vibration energy is transferred completely from one pendulum to the other. By the symmetry of the system we see that the process continues. The vibration energy slowly flows back and forth between a and b. One complete round trip for the energy from a to b and back to a is a beat. The beat period is the time for the round trip and is the inverse of the beat frequency.

All of this is predicted by Eqs. (91) and (92). Using $\omega_1 = \omega_{av} + \omega_{mod}$ and $\omega_2 = \omega_{av} - \omega_{mod}$ in Eqs. (91), we get the "almost harmonic" oscillations

$$\begin{aligned}\psi_a(t) &= A \cos (\omega_{av} + \omega_{mod})t + A \cos (\omega_{av} - \omega_{mod})t \\ &= (2A \cos \omega_{mod} t) \cos \omega_{av} t \\ &\equiv A_{mod}(t) \cos \omega_{av} t\end{aligned} \tag{93}$$

and

$$\begin{aligned}\psi_b(t) &= A \cos (\omega_{av} + \omega_{mod})t - A \cos (\omega_{av} - \omega_{mod})t \\ &= (2A \sin \omega_{mod} t) \sin \omega_{av} t \\ &\equiv B_{mod}(t) \sin \omega_{av} t.\end{aligned} \tag{94}$$

Let us find an expression for the energy (kinetic plus potential) of each pendulum. We think of the oscillation amplitude $A_{mod}(t)$ as essentially constant over one cycle of the "fast" oscillation, and we also neglect the energy that is transferred between the weak coupling spring and the pendulum. (If the spring is very weak, it never has a significant amount of stored energy.) Thus during one fast oscillation cycle we think of pendulum a as a harmonic oscillator of frequency ω_{av} with constant amplitude,

A_{mod}. The energy is then easily seen to be given by twice the average value of the kinetic energy (averaged over one "fast" cycle). This gives

$$E_a = \tfrac{1}{2}M\omega_{\text{av}}{}^2 A_{\text{mod}}{}^2 = 2MA^2\omega_{\text{av}}{}^2 \cos^2 \omega_{\text{mod}}t. \tag{95}$$

Similarly,

$$E_b = \tfrac{1}{2}M\omega_{\text{av}}{}^2 B_{\text{mod}}{}^2 = 2MA^2\omega_{\text{av}}{}^2 \sin^2 \omega_{\text{mod}}t. \tag{96}$$

The total energy of both pendulums is constant, as we see by adding Eqs. (95) and (96):

$$E_a + E_b = (2MA^2\omega_{\text{av}}{}^2) = E. \tag{97}$$

The energy difference between the two pendulums is

$$\begin{aligned} E_a - E_b &= E(\cos^2 \omega_{\text{mod}}t - \sin^2 \omega_{\text{mod}}t) \\ &= E \cos 2\omega_{\text{mod}}t = E \cos (\omega_1 - \omega_2)t. \end{aligned} \tag{98}$$

Combining Eqs. (97) and (98) gives

$$E_a = \tfrac{1}{2}E[1 + \cos (\omega_1 - \omega_2)t], \tag{99a}$$

$$E_b = \tfrac{1}{2}E[1 - \cos (\omega_1 - \omega_2)t]. \tag{99b}$$

Equations (99) show that the total energy E is constant and that it flows back and forth between the two pendulums at the beat frequency. In Fig. 1.15 we plot $\psi_a(t)$, $\psi_b(t)$, E_a, and E_b.

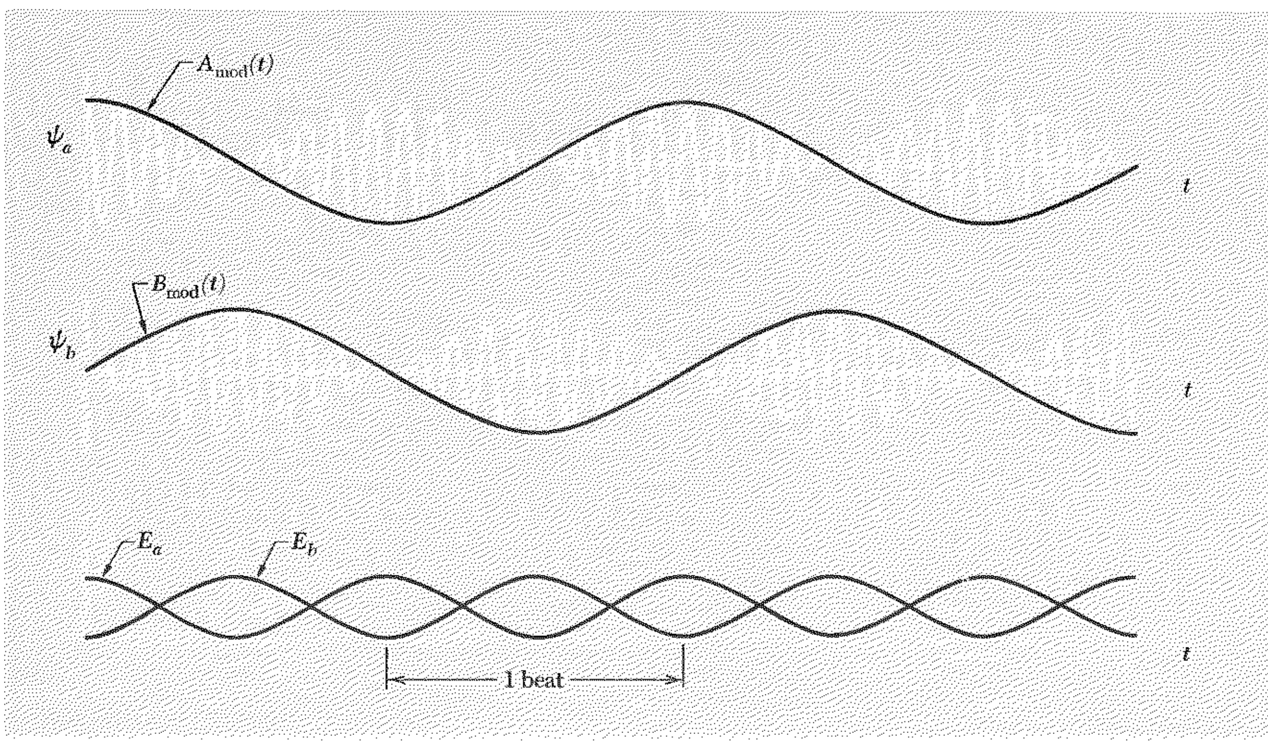


***Fig. 1.15** Energy transfer between two weakly coupled identical pendulums. Energy flows back and forth from a to b at the frequency $|\nu_1 - \nu_2|$, the beat frequency of the two modes.*

Esoteric examples

In the study of microscopic systems—molecules, elementary particles—one encounters several beautiful examples of systems that are mathematically analogous to our mechanical example of two identical weakly coupled pendulums. One needs quantum mechanics to understand these systems. The "stuff" that "flows" back and forth between the two degrees of freedom, in analogy to the energy transfer between two weakly coupled pendulums, is *not* energy but probability. Then energy is "quantized"—it cannot "subdivide" to flow. Either one "moving part" or the other has *all* the energy. What "flows" is the probability to *have* the excitation energy. Two examples, the ammonia molecule (which is the "clockworks" of the ammonia clock) and the neutral K mesons, are discussed in Supplementary Topic 1.

Problems and Home Experiments

1.1 Find the two mode frequencies in cps (cycles per second) for the LC network shown in Fig. 1.12, with $L = 10$ H (henrys) and $C = 6\ \mu$F (microfarads). Also, sketch the current configuration for each mode. *Ans.* $\nu_1 \approx 20$ cps, $\nu_2 \approx 35$ cps.

1.2 If you set a small block of wood (or something) on a record player turntable and look at it from the side as the turntable goes around, using only one eye so as to get rid of your depth perception, the apparent motion (i.e., motion projected perpendicular to your line of sight) is harmonic, i.e., of the form $x = x_0 \cos \omega t$. (*a*) Prove the foregoing statement. (*b*) Make a simple pendulum by suspending a small weight (like a nut or bolt) from a string hung over the back of a chair. Adjust the length of string until you can get your pendulum to swing in synchronization with the projected motion of the block on the turntable when the record player is set at 45 rpm. This gives you a nice demonstration of the fact that the projection of a uniform circular motion is a harmonic oscillation. It is also a nice way to measure g. If g has the standard "textbook value" of 980 cm/sec^2, show that $l \approx 45$ cm for $\nu = 45$ rpm. That should be easy to remember!

Home experiment

1.3 TV set as a stroboscope. The light emitted by a TV set makes a good stroboscope. A given point on the screen is actually dark most of the time; it is lit a small fraction of the time at a regular repetition rate. (You can see this by waving your finger rapidly in front of the screen.) Let us call the regular repetition rate ν_{TV}. The object of this experiment is to measure ν_{TV}. We will tell you that it is either 30 or 60 cps. (For the frequency to be accurately at its proper value, the set should be tuned to a station and locked in on a stable picture—not one that is flickering or drifting.)

(*a*) As a very crude measurement, wave your finger in steady oscillation in front of the screen at a rhythm of about 4 cps, for example. Your finger will block the light from the screen wherever it happens to be when the screen flashes on. Measure the amplitude of your finger's oscillation. Measure the separation between successive