We found that $\nabla \cdot \vec{E} = \rho/\epsilon_0$ can tell us alot of the electric field and we have developed two ways of solving the "Coulomb problem."

① $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}')\,d\tau'}{\mathscr{r}^2}\hat{\mathscr{r}}$ Direct Integration (superposition)

② $\oint_S \vec{E}\cdot d\vec{A} = \int_V \frac{\rho(\vec{r}')\,d\tau'}{\epsilon_0}$ Gauss' Law (works w/ some symmetries)

As we build our theoretical toolbox, we should consider other possible methods to develop solutions. We haven't yet looked into the curl of $\vec{E}$ ($\nabla \times \vec{E}$) to see what it affords us.

Let's remind ourselves of the curl,

$$\nabla \times \vec{v} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$ in Cartesian (for other forms check Griffiths)

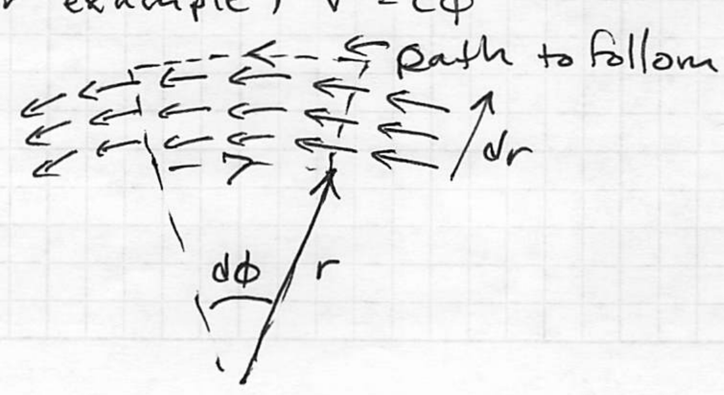
<u>Clicker Question(s)</u>: ① Which of the following have zero curl?
② curl of $\vec{v} = c\hat{\phi}$?

+ Sometimes it can be easy to visualize the curl, e.g. by using the paddle wheel idea (Does it turn?)

+ Another tool is the "circulation" ~~integgat~~. integral

$$\oint_P \vec{v}\cdot d\vec{l} \neq 0 \quad \text{then it has curl!}$$

Take for example, $\vec{v} = c\hat{\phi}$



We can "compute" this integral and show it is nonzero

on the bottom, $\int_P \vec{v}\cdot d\vec{l} = -c r\, d\phi$ (negative b/c $\vec{v}$ + $d\vec{l}$ antiparallel)

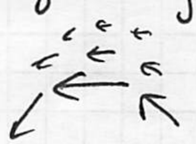
on the sides, $\int_P \vec{v}\cdot d\vec{l} = 0$ b/c $\vec{v} \perp$ to $d\vec{l}$

on the top, $\int_P \vec{v}\cdot d\vec{l} = +c(r+dr)\,d\phi$

so,

$$\oint_P \vec{v}\cdot d\vec{l} = c(r+dr)\,d\phi - c r\, d\phi = c\, dr\, d\phi$$

+ Sometimes, it can be harder to do this by inspection as the fields change in a strange way that might collude.

<u>CQ</u>: what is the curl of  ?

what if we have a description of it? $\vec{v} = \frac{c}{s}\hat{\phi}$

$\vec{v} = \frac{c}{s}\hat{\phi}$ is a type of field that we will encounter in <u>magnetostatics</u>.

In ~~spherical~~ cylindrical

$$\nabla\times\vec{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{z}$$

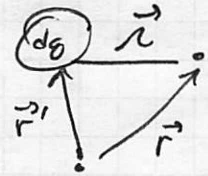
relevant parts of the curl for $\vec{v} = \frac{c}{s}\hat{\phi} = v_\phi\hat{\phi}$,

$$\nabla\times\vec{v} = -\underbrace{\frac{\partial v_\phi}{\partial z}}_{0}\hat{s} + \frac{1}{s}\frac{\partial}{\partial s}(s v_\phi)\hat{z} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{c}{s}\right)\hat{z} = 0\,!$$

<u>So what does $\nabla\times\vec{E}$ do for us?</u>

- It will give us more information about $\vec{E}$.
- It is necessary to know our solutions are unique!
- It will provide another method for solving the "Coulomb Problem".

Well we know that $d\vec{E} = \frac{dq}{4\pi\epsilon_0}\frac{\hat{\imath}}{\imath^2}$



and that we can add up the contributions from a smear of charge to find the total field at $\vec{r}$,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')d\tau'}{\imath^2}\hat{\imath}$$

So, if I wanted to find $\nabla\times\vec{E}$, I could just take the curl, which amounts to finding the curl of,

$$\nabla\times\left[\frac{\vec{r}-\vec{r}\,'}{(\vec{r}-\vec{r}\,')^3}\right]$$

Trust me that we don't want to do that in detail, because it's zero anyway!

We will "prove" this using vector calculus ~~anyway~~ soon, but suffice it to say that in

<u>electrostatics</u>, $\nabla\times\vec{E} = 0$ ($\vec{E}$ cannot be curly in electrostatics)

and this is true for any field $\vec{E}_{net}$ thanks to superposition,

$$\nabla\times\vec{E}_{net} = \nabla\times(\vec{E}_1+\vec{E}_2+\vec{E}_3+\ldots) = \underbrace{\nabla\times\vec{E}_1}_{0} + \underbrace{\nabla\times\vec{E}_2}_{0} + \ldots \rightarrow 0$$

$\nabla\times\vec{E}_{net} = 0$. for any electrostatic field, $\vec{E}_{net}$.

<u>Uniqueness Theorem</u>

As it turns out, the fact that we specify the divergence ($\nabla\cdot\vec{E} = \rho/\epsilon_0$) and the curl ($\nabla\times\vec{E} = 0$) of the field $\vec{E}$ means that for a given setup (problem) $\vec{E}$ is unique (i.e. there is <u>one</u> <u>and</u> <u>only</u> <u>one</u> solution to our problem; we are guarenteed this)!

As we will eventually see this is related to the Helmholtz thm.

Now that we know that $\nabla \times \vec{E} = 0$, we require some additional mathematics to help us unpack the implications of this and to support additional developments that stem from $\nabla \times \vec{E} = 0$.

① Curl of a gradient is zero

For any f, $\nabla \times \nabla f = 0$

Conceptually, ∇f points up the hill, so it's a "radial" like field; it has no curl.

Proof: Consider $(\nabla \times \vec{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$

in this case,

$$(\nabla \times \nabla f)_z = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 0 \checkmark$$

② We know the gradient of the separation vector.

$$\nabla \frac{1}{\imath} = -\frac{\hat{\imath}}{\imath^2}$$

You can prove this by taking the gradient of this function, which you did on homework 1.

So if we go back to our description of the $\vec{E}$-field,

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{r}\,')\, d\tau' \, \frac{\hat{\imath}}{\imath^2}$$ $\Rightarrow$ from the second point above we can replace

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{r}\,')\, d\tau' \left(-\nabla \frac{1}{\imath}\right)$$ $\Leftarrow$ $\frac{\hat{\imath}}{\imath^2}$ with $-\nabla \frac{1}{\imath}$

Clicker Question: Ok to move ∇ out?

$$\vec{E} = -\nabla\left(\frac{1}{4\pi\epsilon_0} \int \rho(\vec{r}\,')\, d\tau' / \imath \right) \equiv -\nabla V(\vec{r})$$

where $V(\vec{r}) \equiv \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')\, d\tau'}{\imath}$

– we have found a function $V(\vec{r})$ that reduces the Coulomb problem from a vector one (find $\vec{E}$) to a scalar one (Find V)

So Because $\vec{E} = -\nabla V$,

then

$$\nabla \times \vec{E} = -\nabla \times (\nabla V) = 0$$ from the first pt. above.

$\nabla \times \vec{E} = 0$, always, <u>in electrostatics</u>

– $\vec{E}$ has no curl.

We have been able to define V interms of ρ,

$$V(\vec{r}) = \int \frac{1}{4\pi\epsilon_0} \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}}$$

and $\vec{E}$ in terms of V,

$$\vec{E} = -\nabla V$$

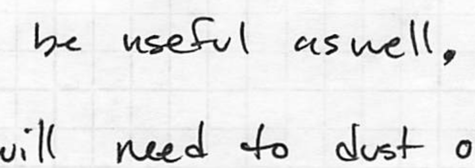
But we are also able to define V in termsof $\vec{E}$, which can be useful aswell,

Here we will need to dust off Stokes theorem,

$$\int_{\text{open } S} (\nabla \times \vec{F}) \cdot d\vec{A} = \oint_C \vec{F} \cdot d\vec{l} \quad (\text{for any } \vec{F})$$

$\nabla \times \vec{F}$ is the circulation or swirl at a point

If we add up all the swirls overa surface S, we get the circulation around the outside (boundary of S).



<u>Clicker Question</u>: if $\nabla \times \vec{E} = 0$, $\oint_C \vec{E} \cdot d\vec{l} = ?$

$$\oint_{\text{any loop}} \vec{E} \cdot d\vec{l} = 0$$ in electrostatics

Finally, let's review the fundamental theorem of calculus – or the gradient theorem in this context as $\vec{E} = -\nabla V$.

$$\int_A^B (\nabla F)\cdot d\vec{l} = F(B) - F(A) \qquad \text{b/c } \vec{E} = -\nabla V,$$

$$-\int_A^B \vec{E}\cdot d\vec{l} = \int_A^B \nabla V \cdot d\vec{l} = V(B) - V(A)$$

So we can find $V(r)$ in terms of $\vec{E}$,

$$V(\vec{r}_2) - V(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}(\vec{r})\cdot d\vec{l}$$

<u>So what's V?</u>

- V is a scalar function that we call the "potential" or the "electric potential" (It is not the potential energy!)
- It's a scalar field; there's a # at every point in space that defines V.

For a point charge, $V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$ B/c $\rho \to q\delta^3(r)$

- But there is some ambiguity with V, we can always add a constant to it and no affect the calculation of $\vec{E}$ *

$$V \to V' = V + C \Rightarrow \vec{E} = -\nabla V = -\nabla V'$$

Because $\nabla C = 0$.

- We typically set $V(\vec{r}\to\infty) = 0$ to set the value of the constant C *

* Note! You must be careful here b/c sometimes $V \not\to 0$ as $r \to \infty$.

Clicker Question: Can superposition be used w/ V? *

* Yes, mostly fine, but all V's must have the same zero!

By superposition, $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')d\tau'}{\mathscr{r}}$ with some $\rho(\vec{r}')$

Summary of what we have found so far

We can reduce our vector problem to a scalar one by using V! Greatly simplified

$$V(\vec{r}) = -\int_A^B \vec{E}\cdot d\vec{l} \longleftarrow \text{Find } V \text{ from } \vec{E}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}')d\tau'}{\mathscr{r}} \leftarrow \text{Find } V \text{ from } \rho$$

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r}) \longleftarrow \text{Find } \vec{E} \text{ from } V$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}')d\tau'}{(\vec{r}-\vec{r}')^3}(\vec{r}-\vec{r}') \leftarrow \text{Find } \vec{E} \text{ from } \rho$$

The integral for V is often easier, so we might prefer to find V & then use $\vec{E} = -\nabla V$ to find $\vec{E}$.

- So V & $\vec{E}$ are intimately connected, but we do need to be careful about what we can conclude about $\vec{E}$ from V and vice-versa.

Clicker Questions: Potential is X → E is?

Punchline: the value of the potential doesn't tell you about $\vec{E}$, how it changes does.

Now that we have $\vec{E} = -\nabla V$, what does $\nabla \cdot \vec{E}$ tell us about V?

$$\nabla \cdot \vec{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V \quad \left(\text{Laplacian of } V.\right)$$

this tell us that,

$$\nabla^2 V = -\rho/\epsilon_0$$

this is Poisson's Equation (it's really a combination of Gauss & $\nabla \times \vec{E} = 0$)

And in fact, this is all we need. Given ρ, find V, then get $\vec{E}$

If we are in empty space,

$$\nabla^2 V = 0$$

this is Laplace's Equation. We will spend a lot of time solving this soon (lots of new math to unpack later).

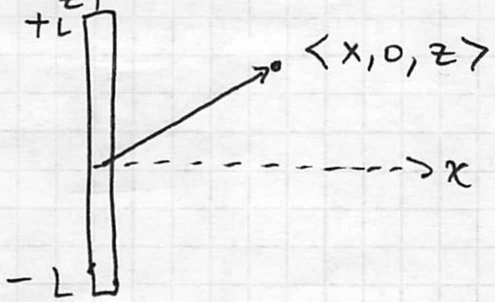
Examples

- the case of a pt. charge is one we have already encountered,

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

If you have several charges, use superposition ($V_1 + V_2 + \ldots$)

- What about a line charge with uniform density λ? Let's find V at $\langle x, 0, z\rangle$ as shown.



finite length, L

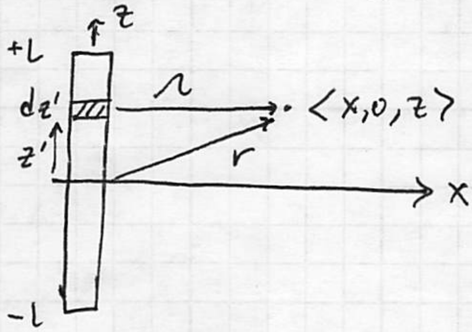
We want $\vec{E}(\vec{r})$, but

- Gauss' Law is no good (no good symmetry, no obvious surface)

We could integrate to find $\vec{E}$,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')\,d\tau'}{(\vec{r}-\vec{r}\,')^3}(\vec{r}-\vec{r}\,')$$

But let's use V instead and then compute $\vec{E} = -\nabla V$



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}\,')\,d\tau'}{\mathscr{r}}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_P \frac{\lambda(\vec{r}\,')\,dl'}{\mathscr{r}}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{-L}^{L} \frac{\lambda\,dz'}{\sqrt{x^2+(z-z')^2}} = \frac{\lambda}{4\pi\epsilon_0}\underbrace{\int_{-L}^{L} \frac{dz'}{\sqrt{x^2+(z-z')^2}}}$$

$$V(\vec{r}) = \frac{\lambda}{4\pi\epsilon_0}\log\left[\frac{L+z+\sqrt{x^2+(L+z)^2}}{+L-z+\sqrt{x^2+(L-z)^2}}\right] \longleftarrow$$

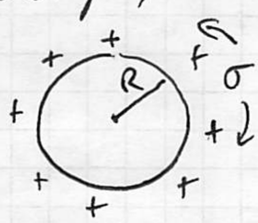
How do we do this? (meh. lookitup)

we could add a constant C, $V \to 0$ as $z, x \to \infty$.

Now take $\frac{\partial}{\partial x} \to E_x$ & $\frac{\partial}{\partial z} \to E_z$

Notice we solved in the $(x-z)$ plane, so no E_y.

As our last example let's go back to the shell of Charge, σ over the surface, radius R



Here, $r > R$: $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$

where $q = \sigma 4\pi R^2$

<u>Clicker Question</u>: what's $\vec{E}$ $r<R$?

$\vec{E} = 0$ $r<R$ so, what's $V(r>R)$?

Define $V(r\to\infty) = 0$ so that,

$$\underset{-0}{V(r>R)} = -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\,dr' = \frac{-q}{4\pi\epsilon_0}\left(-\frac{1}{r'}\right)_{\infty}^{r}$$

$$V(r>R) - \underbrace{0}_{r\to\infty} = \frac{-q}{4\pi\epsilon_0}\left(-\frac{1}{r'}\right)\Big|_{\infty}^{r} = +\frac{q}{4\pi\epsilon_0}\frac{1}{r}$$

what about for $r<R$.?

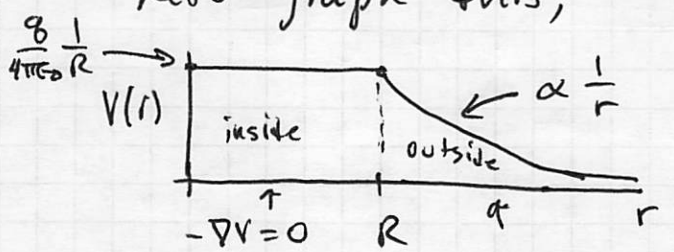
We need a reference pt so ~~whe~~ we will use $V(R)$ where $V(R) = \frac{q}{4\pi\epsilon_0}\frac{1}{R}$,

$$V(r<R) - V(R) = -\int_R^r \vec{E}_{inside}\cdot d\vec{r}' = -\int_R^r 0\, dr' = 0$$

+ that doesn't mean that $V(r<R) = 0$!
+ $V(r<R) = V(R)$ b/c what we found is the potential difference between $V(r<R)$ & $V(R)$.

$$V(r<R) = V(R) = \frac{q}{4\pi\epsilon_0}\frac{1}{R}$$

Let's graph this,



$-\nabla V = 0$ as it should be.

$-\nabla V \propto -\nabla \frac{1}{r} \propto \frac{1}{r^2}$ as it should be

* Two things to notice:
  V is continuous, but slope isn't.

Clicker Question: graph of E or V or neither?

+ $\vec{E}$ can be discontinuous (remember the plane of charge?)
+ V cannot, but its slope can be. Why? $\vec{E} = -\nabla V$.