LINEARIZATION & FIXED PTS.

While we could integrate to find the trajectories we are interested in, we would only find a solution for a given initial condition.

Phase Space

We found with phase space that we could see families of trajectories. that's a useful tool for finding possible areas (parameters) of interest.

Another Tool: Linearization

While it might seem that you should just always use a computer, there's some really good reasons to also learn mathematical tools that can provide additional info and checks.

Linerization of a differential eq. near a fixed pt.

We use linearization to find qualitatively correct but approximate dynamics near fixed pts.

Consider the pair of 1^{st} order ODEs,

$$\dot{x} = f(x,y) \qquad \dot{y} = g(x,y)$$

* assume there's a fixed pt, $\dot{x} = 0$; $\dot{y} = 0$ @ $\langle x^*, y^* \rangle$

thus

$$f(x^*, y^*) = 0 \quad \& \quad g(x^*, y^*) = 0$$

* Let u & v represent small disturbances from x^* & y^*,

$$u = x - x^* \qquad v = y - y^*$$

Now we see if the disturbances grow or decay.

$$\dot{u} = \frac{d}{dt}(x - x^*) = \dot{x} \qquad (x^* \text{ constant})$$

$$\dot{u} = f(x^* + u,\ y^* + v)$$

Expand w/ Taylor

$$\dot{u} = \underbrace{f(x^*, y^*)}_{\substack{= 0 \\ \text{by defn} \\ \text{of} \\ \text{fixed pt.}}} + u \frac{\partial f}{\partial x}\bigg|_{x^*, y^*} + v \frac{\partial f}{\partial y}\bigg|_{x^*, y^*} + O(u^2, v^2, uv)$$

linear in u & v

$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ evaluated @ fixed pt $\langle x^*, y^* \rangle$

$$\dot{u} = u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + O(u^2, v^2, uv)$$

By analogy,

$$\dot{v} = u \frac{\partial g}{\partial x} + v \frac{\partial g}{\partial y} + O(u^2, v^2, uv)$$

Two general Diffy Qs, for disturbances!

$$\boxed{\begin{aligned} \dot{u} &= u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv) \\ \dot{v} &= u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + O(u^2, v^2, uv) \end{aligned}}$$

We can rewrite them,

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}}_{} \begin{bmatrix} u \\ v \end{bmatrix} + \text{quad terms}$$

$$\underline{\underline{A}} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{(x^*, y^*)}$$

The Jacobian evaluated @ fixed pt

The Linearized System

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\dot{\vec{x}} = A\vec{x} \quad \text{where } \vec{x} = \begin{bmatrix} u \\ v \end{bmatrix}$$