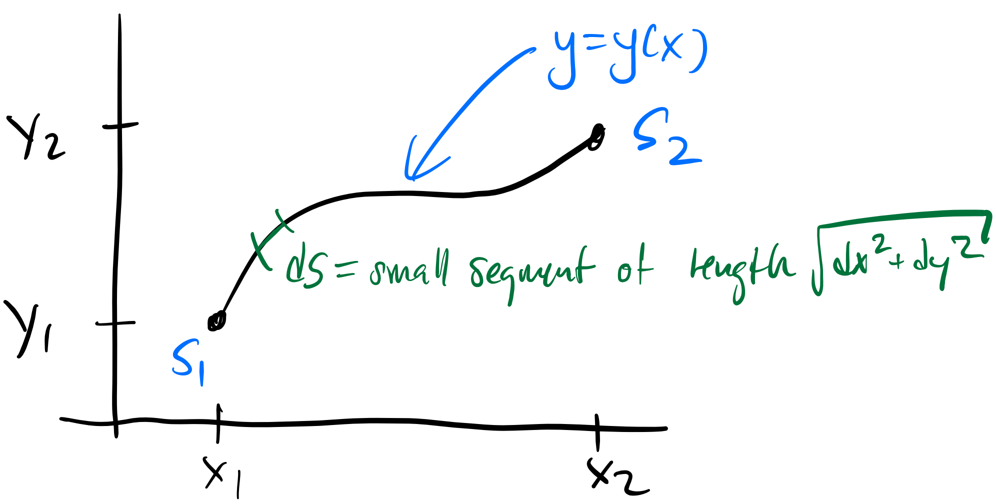
Calculus of Variations

- as we will use it the calculus of variations focuses on finding (the conditions of) extrema for quantities that can be expressed as an integral.
- this might seem odd but turns out to be an interesting way to develop an equivalent formulation of mechanics.

Canonical Conceptualization

Using Calculus of variations we can show the shortest distance between two points is a line.

Consider a general path in 2D



The length of the path is the integral from S_1 to S_2

$$\ell = \int_{S_1}^{S_2} ds$$

← we want to minimize ℓ, which is minimizing the integral.

Let's write ds in terms of dx, dy, and $y(x)$.

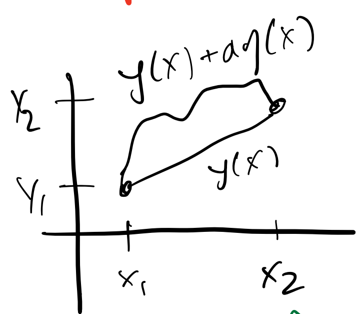
$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2} = dx\sqrt{1+y'(x)^2}$$

So that,

$$\ell = \int_{x_1}^{x_2} dx \sqrt{1+\underbrace{[y'(x)]^2}_{}}$$

$y'(x)$ defines the path

To go further, we need to posit an incorrect solution,



$$Y(x) = \underbrace{y(x)}_{\text{correct}} + \underbrace{\alpha\eta(x)}_{\text{error to minimize.}}$$

$$Y(x_1) = Y_1 \qquad Y(x_2) = Y_2$$

And we need to investigate what happens in general,

Take a function that is being integrated,

$$f(y(x), y'(x), x)$$

$$S = \int_{x_1}^{x_2} f(y(x), y'(x), x)\, dx$$

assume $y(x)$ minimizes S such that any function $Y(x) = y(x) + \underbrace{\alpha \eta(x)}_{\text{error term}}$ produces a larger integral,

$$\int_{x_1}^{x_2} f(Y(x), Y'(x), x)\, dx > \int_{x_1}^{x_2} f(y(x), y'(x), x)\, dx$$

Thus $S(\alpha = 0)$ is a minimum, what conditions does that produce?

$$\left.\frac{dS}{d\alpha}\right|_{\alpha=0} = 0$$

finds the extrema

Start Long Derivation

$$S = \int_{x_1}^{x_2} f(Y(x), Y'(x), x)\,dx$$

$$S = \int_{x_1}^{x_2} f\left(y(x) + \alpha\eta(x), y'(x) + \alpha\eta'(x), x\right)dx$$

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \frac{d}{d\alpha}\left[f\left(y(x)+\alpha\eta(x), y'(x)+\alpha\eta'(x), x\right)\right] dx$$

$$= \int_{x_1}^{x_2} \left[\frac{df}{dY}\underbrace{\frac{dY}{d\alpha}} + \frac{df}{dY'}\underbrace{\frac{dY'}{d\alpha}} + \frac{df}{dx}\underbrace{\frac{dx}{d\alpha}}_{0}\right] dx$$

$$\frac{dY}{d\alpha} = \frac{d}{d\alpha}(y + \alpha\eta) = \eta$$

$$\frac{dY'}{d\alpha} = \frac{d}{d\alpha}(y' + \alpha\eta') = \eta'$$

$$\frac{df}{dY} = \frac{df}{dY}\frac{dy}{dy} = \frac{df}{dy}\left[\frac{dY}{dy}\right] \rightarrow 1$$

why? const in y

$$Y(x) = y(x) + \alpha\eta(x)$$

$$\frac{dY}{dy} = 1$$

$$\frac{df}{dY'} = \frac{df}{dy'}$$ same reason

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \left[\frac{df}{dy} y + \frac{df}{dy'} y' \right] dx$$

Set the integral to zero, $\left.\frac{dS}{d\alpha}\right|_{\alpha=0} = 0$

$$\int_{x_1}^{x_2} \left(y \frac{df}{dy} + y' \frac{df}{dy'} \right) dx = 0$$

Integrate by parts:

$$\int u'v \, dx = \underbrace{[uv]} - \int uv' \, dx$$

apply to 2nd term

"surface term" evaluated at x_1, x_2

$$\int_{x_1}^{x_2} y' \frac{df}{dy'} dx = \underbrace{\left[y \frac{df}{dy'} \right]_{x_1}^{x_2}}_{y(x_2) = y(x_1) = 0} - \int_{x_1}^{x_2} y \frac{d}{dx}\left(\frac{df}{dy'} \right) dx$$

Surface term vanishes

So,

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \left(y \frac{df}{dy} - y \frac{d}{dx}\left(\frac{df}{dy'} \right) \right) dx = 0$$

ok after all that,

$$\frac{dS}{d\alpha} = \underbrace{\int_{x_1}^{x_2} y(x) \left[\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right) \right] dx}_{\text{must be true for any } y(x)} = 0$$

must be true for any $y(x)$ so,

$$\boxed{\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right) = 0}$$

Euler - Lagrange eqn for 1D

Return to our line problem

$$l = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$$

here,

$$f(y, y', x) = \sqrt{1+y'^2}$$

now let's apply the Euler-Lagrange formulation

$$\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right) = 0$$

$$\frac{df}{dy} = 0 \qquad \frac{df}{dy'} = \frac{1}{2}\left(1+y'^2\right)^{-1/2}(2y')$$

$$\frac{df}{dy'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$-\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$y'(x)$ function of x purely so

$$\boxed{\frac{y'}{\sqrt{1+y'^2}} = \text{constant}}$$

or,

$$y' = c\sqrt{1+y'^2}$$

$$y'^2 = c^2(1+y'^2)$$

$$y'^2(1-c^2) = c^2$$

$$y'^2 = c^2/(1-c^2)$$

So,

$$y' = \sqrt{\frac{c^2}{1-c^2}} = \text{some other constant} = m$$

$$\frac{dy}{dx} = m = \text{const} \Rightarrow \boxed{y(x) = mx + b}$$