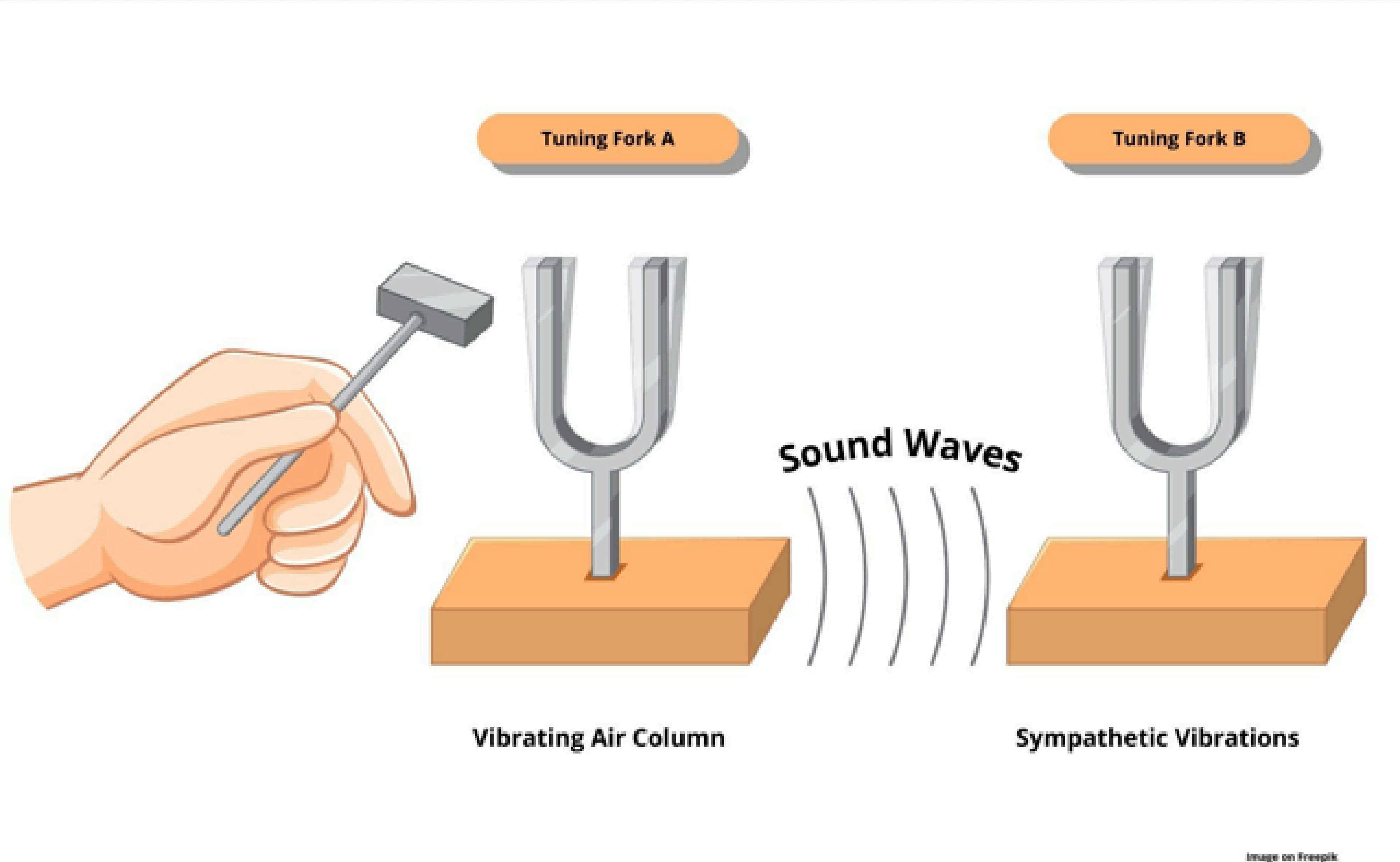
Day 25 - Resonance



Announcements

- Midterm 1 is still being graded
- Homework 6 is due Friday
- Homework 7 is posted, due next Friday
- No office hours today

Seminars this week

Most of MSU folks are at APS Global Physics Summit

WEDNESDAY, March 19, 2025

- Astronomy Seminar, 1:30 pm, 1400 BPS, Alex Rodriguez, University of Michigan, *Galaxy clusters, cosmology, and velocity dispersion*
- FRIB Nuclear Science Seminar, 3:30pm., FRIB 1300 Auditorium, Dr. Pierre Morfouace of CEA-DAM, *Mapping the new asymmetric fission island with the R3B/SOFIA setup*

THURSDAY, March 20, 2025

Colloquium, 3:30 pm, 1415 BPS, Guillaume Pignol, University of Grenoble, *Ultracold neutrons: a precision tool in fundamental physics*

Reminders

We started to solve the forced harmonic oscillator equation:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$$

We examined the case of a sinusoidal driving force:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$$

There's a complimentary case where the driving force is a sine wave:

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = f_0 \sin(\omega t)$$

Reminders

We combined the two equations into a complex equation using these identities:

$$z(t) = x(t) + iy(t)$$

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

The resulting equation is:

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$$

Notice that there's a homogeneous part (z_h) and a particular part (z_p).

$$\ddot{z}_h + 2\beta\dot{z}_h + \omega_0^2 z_h = 0$$

Reminders

The homogeneous part is the solution we've found before with the general solution:

$$z_h(t) = C_1 e^{rt} + C_2 e^{r^* t}$$

where $r = -\beta \pm i\sqrt{\omega_0^2 - \beta^2}$. In the case of a weakly damped oscillator ($\beta^2 < \omega_0^2$), we have:

$$z_h(t) = e^{-\beta t}\left(C_1 e^{-i\sqrt{\omega_0^2-\beta^2}t} + C_2 e^{+i\sqrt{\omega_0^2-\beta^2}t}\right)$$

These solutions die out as $t \to \infty$. They are called **transient solutions**.

Solving the particular part

The particular part is the solution to the driven harmonic oscillator equation:

$$\ddot{z}_p + 2\beta\dot{z}_p + \omega_0^2 z_p = f_0 e^{i\omega t}$$

Assume a sinusoidal solution (frequency, ω) of the form:

$$z_p(t) = Ce^{i\omega t}$$

where C is a complex number. Then, we have:

$$-\omega^2 Ce^{i\omega t} + 2i\beta\omega Ce^{i\omega t} + \omega_0^2 Ce^{i\omega t} = f_0 e^{i\omega t}$$
$$\left(-\omega^2 + 2i\beta\omega + \omega_0^2\right)C = f_0$$

Amplitude of the particular solution

$$C = \frac{f_0}{\left(\omega_0^2 - \omega^2 + 2i\beta\omega\right)}$$

We want to convert this to polar form:

$$C = Ae^{-i\delta}$$

where A and δ are real numbers. We use the complex form to compute the magnitude of the amplitude:

$$A^2 = C\bar{C} = \frac{f_0^2}{\left(\omega_0^2 - \omega^2 + 2i\beta\omega\right)\left(\omega_0^2 - \omega^2 - 2i\beta\omega\right)}$$

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

Clicker Question 24-5

We found that the square amplitude of the driven harmonic oscillator is:

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

When is the amplitude of the driven oscillator maximized?

1. When the driving frequency (ω) is far from the natural frequency (ω_0)
2. When the driving frequency (ω) is close to the natural frequency (ω_0)
3. When the damping (2β) is weak
4. When the damping (2β) is strong
5. Some combination of the above

Finding the phase

With,

$$C = \frac{f_0}{\left(\omega_0^2 - \omega^2 + 2i\beta\omega\right)} = Ae^{-i\delta}$$

then we can compare the complex forms:

$$f_0 e^{i\delta} = A\left(\omega_0^2 - \omega^2 + 2i\beta\omega\right).$$

Both f_0 and A are real numbers, so the phase δ is the same phase as the complex number:

$$\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

The Particular Solution

Let's return to the particular solution:

$$z_p(t) = Ce^{i\omega t} = Ae^{-i\delta}e^{i\omega t} = Ae^{i(\omega t - \delta)}$$

$$z_p(t) = A\cos(\omega t - \delta) + iA\sin(\omega t - \delta)$$

So we get solutions to both driven oscillators:

$$x_p(t) = Re(z_p(t)) = A\cos(\omega t - \delta)$$

$$y_p(t) = Im(z_p(t)) = A\sin(\omega t - \delta)$$

These are the **steady-state solutions**.

They persist as $t \to \infty$ and oscillate at the driving frequency ω.

The Full Solution

$$x(t) = x_h(t) + x_p(t)$$

Here, $x_h(t)$ is the transient solution and $x_p(t)$ is the steady-state solution.

For weakly damped oscillators, the transient solution can be written in the form:

$$x_h(t) = A_{tr} e^{-\beta t} \cos(\sqrt{\omega_0^2 - \beta^2} t + \delta_{tr})$$

where A_{tr} and δ_{tr} are real numbers and are the amplitude and phase of the transient solution. Both are determined by the initial conditions.

$$x_p(t) = A \cos(\omega t - \delta)$$

where A and δ are real numbers and are the amplitude and phase of the steady-state solution.

The Full Solution

The transient plus the steady-state solution is the full solution:

$$x(t) = A_{tr} e^{-\beta t} \cos(\sqrt{\omega_0^2 - \beta^2} t + \delta_{tr}) + A \cos(\omega t - \delta)$$

As $t \to \infty$, the transient solution dies out and the steady-state solution persists.

$$x(t \to \infty) = A \cos(\omega t - \delta)$$

where

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

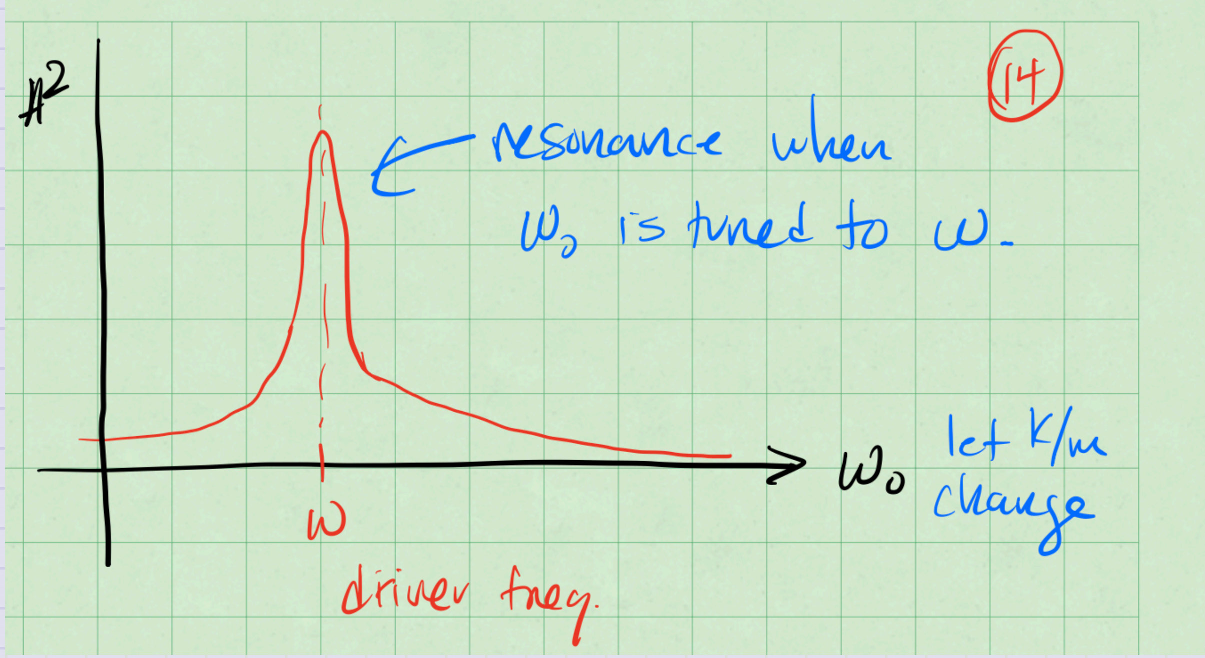
$$\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

Resonance

The amplitude of the steady-state solution is:

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

We change ω_0 and observe how the amplitude changes.



Achieving resonance

The denominator of the equation controls the amplitude:

$$(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2$$

Case 1: Tune ω_0 to be close to ω. *Car Radio tuning*

With $\omega_0 = \omega$, the amplitude is:

$$A^2 = \frac{f_0^2}{4\beta^2\omega^2}$$

Achieving resonance

The denominator of the equation controls the amplitude:

$$(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2$$

Case 2: Tune ω to be close to ω_0. *Pushing a swing*

Find the ω that maximizes the amplitude by taking the derivative with respect to ω:

$$\frac{d}{d\omega}\left((\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\right) = 0$$

$$2(\omega_0^2 - \omega^2)(-2\omega) + 8\beta^2\omega = 0$$

$$4\omega(\omega^2 - \omega_0^2 + 2\beta^2) = 0$$

$$\omega = 0 \qquad \omega = \sqrt{\omega_0^2 - 2\beta^2}$$