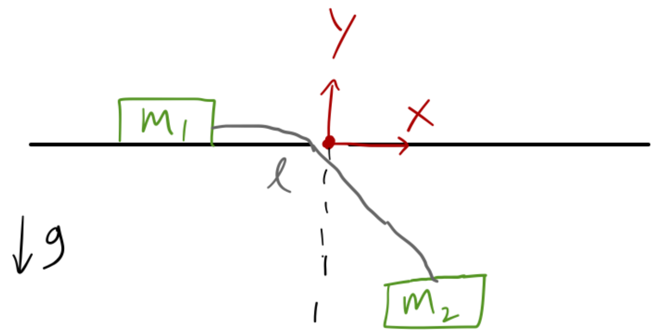
Tutorial: Lagrangian Mechanics



Two masses m_1 and m_2 are connected by a massless rope of length ℓ. As pictured, one of them rests on a frictionless table, while the other is dangling over the edge. We'll treat this as a **two-dimensional** system, starting with Cartesian coordinates (x, y) as depicted. Note that in particular, this means that **block 2 is free to swing back and forth!**

I. Choosing coordinates

A. Using the Cartesian coordinates of the two blocks (x_1, y_1) and (x_2, y_2), write down a set of equations which represent all of the constraints on the system as given.

B. How many degrees of freedom does this system have?

C. Sketch a set of generalized coordinates on the diagram above.

D. [Discussion] By treating this problem as two-dimensional, we are *ignoring* the coordinates z_1 and z_2, knowing that if $z_1(0) = z_2(0) = 0$, the system doesn't move in the z-direction. What about the coordinate x_2 - is it *ignorable*? Why or why not?

II. Setting up the Lagrangian

On the board, we've set up a common set of generalized coordinates (GCs) to use, and re-expressed the Cartesian coordinates using them. Let's go on to write down the Lagrangian describing this system.

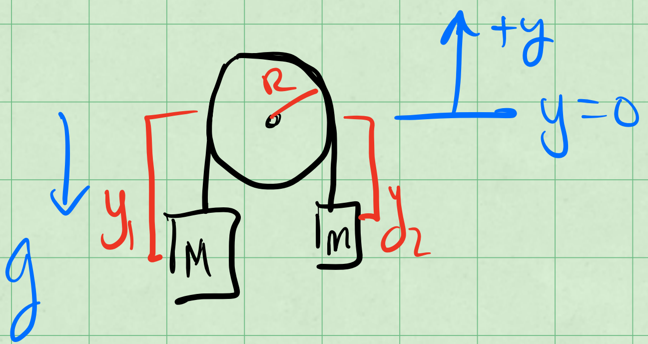
A. Write the potential energy U in terms of the generalized coordinates.

B. Evaluate the time derivatives of the Cartesian coordinates in terms of the GCs (and their derivatives), and use your results to write the kinetic energy T.

C. Combine your answers from A and B to obtain the Lagrangian, $\mathcal{L}$.

D. *[Discussion]* Find the equations of motion for both GCs by applying the Euler-Lagrange equations. What do you notice about the equations of motion for the two GCs? What does this tell you about the motion of the system?

Example: Atwood's Machine



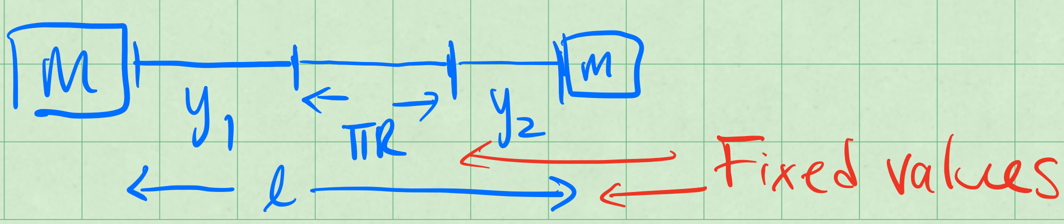
Let's do a naive application with y_1 & y_2

$$U = + mgy_1 + mgy_2$$

$$T = \frac{1}{2} M\dot{y}_1^2 + \frac{1}{2} m\dot{y}_2$$

But! y_1 & y_2 are not independent!

Unravel the string



so there's an equation of constraint, ②

$$\ell = y_1 + \pi R + y_2 \implies y_2 = (\ell - \pi R) - y_1$$

this constraint has implications for velocities,

$$\frac{dy_2}{dt} = \frac{d}{dt}\left([\ell - \pi R] - y_1 \right) = -\frac{dy_1}{dt}$$

$$\underline{\dot{y}_2 = -\dot{y}_1} \quad \underline{\text{as expected!}}$$

Let's use this constraint to take

$$T(\dot{y}_1, \dot{y}_2) = T(\dot{y}_1)$$

$$U(y_1, y_2) = U(y_1)$$

$$T = \frac{1}{2} M \dot{y}_1^2 + \frac{1}{2} m \dot{y}_2^2 = \frac{1}{2}(M+m)\dot{y}_1^2$$

$$U = +Mgy_1 + mgy_2$$

$$U = Mgy_1 + mg([\ell - \pi R] - y_1)$$

$$U = (M-m)gy_1 + \underbrace{mg[\ell - \pi R]}_{U_0 \text{ a constant}}$$

Dropped

$$\mathcal{L} = T - U$$

$$\mathcal{L} = \frac{1}{2}(M+m)\dot{y}_1^2 - (M-m)gy_1$$

y_1:

$$\frac{d\mathcal{L}}{dy_1} - \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{y}_1}\right) = 0$$

$$-(M-m)g - \frac{d}{dt}\left((M+m)\dot{y}_1\right) = 0$$

$$F_{y_1} = -(M-m)g$$ (net force on M_1)

$$F_{x_i} = \frac{d\mathcal{L}}{dx_i}$$

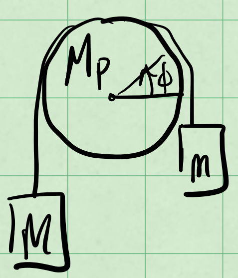
$$(M+m)\ddot{y}_1 = -(M-m)g$$

$$\boxed{\ddot{y}_1 = -\frac{(M-m)}{(M+m)}g}$$

But wait there's more!

$\Rightarrow$ What if the pulley rotates?

$\Rightarrow$ The rope cannot slip, so that a rotation $Rd\phi$ gives a change dy_1 exactly. (no slip constraint)



We introduce

$$T_{pulley} = \frac{1}{2} I\omega^2$$

$$T_p = \frac{1}{2} M_p R^2 \dot{\phi}^2$$

We now map this new kinetic energy into the problem.

$$T(\dot{y}_1, \dot{\phi}) = \frac{1}{2}(M+m)\dot{y}_1^2 + \frac{1}{2}M_p R^2 \dot{\phi}^2$$

But the constraint is such that,

$$dy_1 = R\,d\phi \implies y_1 = R\phi + \underbrace{R\phi_0}_{\text{a constant}}$$

$$\implies \dot{y}_1 = R\dot{\phi}$$

Work these back into T, U, & $\mathcal{L}$,

$$T = \frac{1}{2}(M+m)\dot{y}_1^2 + \frac{1}{2}M_p R^2 \dot{\phi}^2 = \frac{1}{2}(M+m+M_p)R^2\dot{\phi}^2$$

$$U = (M-m)g y_1 + U_0 = (M-m)gR\phi + \underbrace{\tilde{U}_0}_{\text{new constant, discard}}$$

$$\mathcal{L}(\phi, \dot{\phi}) = T(\dot{\phi}) - U(\phi)$$

$$\mathcal{L} = \frac{1}{2}(M_T)R^2\dot{\phi}^2 - (M-m)gR\phi$$

ϕ:

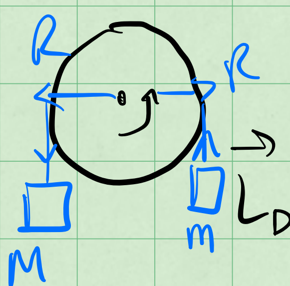
$$\frac{d\mathcal{L}}{d\phi} - \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\phi}}\right) = 0$$

$$-(M-m)gR - \frac{d}{dt}\left(M_T R^2 \dot{\phi}\right) = 0$$

$$F_\phi = \frac{d\mathcal{L}}{d\phi} = \underbrace{-(M-m)gR}_{\text{a torque}} = |\vec{r} \times \vec{F}_{net,y}|$$

$$P_\phi = M_T R^2 \dot{\phi} \Rightarrow |\vec{L}|$$

angular momentum of whole "system" about axle



$$\vec{L}_{Disk} + \vec{L}_M + \vec{L}_m = \vec{L}_{TOT}$$

$\vec{L}_{Disk} = I\vec{\omega} = M_D R^2 \dot{\phi}$ (out of page)

$\vec{L}_M = \vec{r}_M \times \vec{p}_M = (R)(MR\dot{\phi})$ (out of page)

$\vec{L}_m = \vec{r}_m \times \vec{p}_m = (R)(mR\dot{\phi})$ " "

add them to get $\vec{L}_{TOT}$

$$\left|\vec{L}_{tot}\right| = M_T R^2 \dot{\Phi} = P_\phi \quad \text{above}$$

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Back to the Equation of Motion

$$-(M-m)gR - \frac{d}{dt}\left(M_T R^2 \dot{\Phi}\right) = 0$$

$$-(M-m)gR - M_T R^2 \ddot{\Phi} = 0$$

$$\boxed{\ddot{\Phi} = -\frac{g}{R}\,\frac{M_T}{(M-m)}}$$

Constant acceleration